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BOUNDEDNESS ON MORREY SPACE FOR TOEPLITZ TYPE OPERATOR ASSOCIATED TO SINGULAR INTEGRAL OPERATOR WITH VARIABLE CALDERÓN–ZYGMUND KERNEL

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Abstract. In this paper, the boundedness of the Toeplitz type operators associated to the singular integral operator with variable Calderón-Zygmund kernel on Morrey spaces is obtained. For this purpose, some M^k-type sharp maximal function inequalities for the operators are proved.

1. Introduction and Preliminaries

As the development of singular integral operators (see [7], [21]), their commutators have been well studied. In [4], [19], [20], the authors proved that the commutators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [2]) proved a similar result when singular integral operators are replaced by the fractional integral operators. In [1], Calderón and Zygmund introduced some singular integral operators with variable kernel and discussed their boundedness. In [11–13], [22], the authors obtained the boundedness for the commutators generated by the singular integral operators with variable kernel and BMO functions. In [15], the authors proved the boundedness for the multilinear oscillatory singular integral operators generated by the operators and BMO functions. In [8], [9], [14], some Toeplitz type operators associated to the singular integral operators and strongly singular integral operators are introduced, and the boundedness for the operators generated by BMO and Lipschitz functions is obtained. Motivated by these, in this paper, we will study the Toeplitz type operator generated by the singular integral operator with variable Calderón-Zygmund kernel.

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$f^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

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where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)dx$. It is well-known that (see [7], [21])

$$f^{\#}(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

We say that f belongs to $BMO(R^n)$ if $f^{\#}$ belongs to $L^{\infty}(R^n)$ and define $||f||_{BMO} = ||f^{\#}||_{L^{\infty}}$. It is known that (see [21])

$$||f - f_{2^k Q}||_{BMO} \leqslant Ck||f||_{BMO}.$$

For $0 < r < \infty$, we denote $f_r^{\#}$ by

$$f_r^{\#}(x) = [(|f|^r)^{\#}(x)]^{1/r}.$$

Let

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

For $\eta > 0$, let $M_{\eta}(f) = M(|f|^{\eta})^{1/\eta}$. For $k \in N$, we denote by M^k the operator M iterated k times, i.e., $M^1(f) = M(f)$ and

$$M^k(f) = M(M^{k-1}(f)) \text{ when } k \geqslant 2.$$

For $0 < \eta < n$ and $1 \leqslant r < \infty$, set

$$M_{\eta,r}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-r\eta/n}} \int_Q |f(y)|^r dy \right)^{1/r}.$$

Let Φ be a Young function and $\tilde{\Phi}$ be the complementary associated to Φ, we denote the Φ-average for a function f by

$$||f||_{\Phi,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi \left(\frac{|f(y)|}{\lambda} \right) dy \leqslant 1 \right\}$$

and the maximal function associated to Φ by

$$M_{\Phi}(f)(x) = \sup_{x \in Q} ||f||_{\Phi,Q}.$$

The Young functions which we use in this paper are $\Phi(t) = t(1 + logt)$ and $\tilde{\Phi}(t) = exp(t)$, the corresponding average and maximal functions are denoted by $|| \cdot ||_{L(logL),Q}$, $M_{L(logL)}$ and $|| \cdot ||_{expL,Q}$, M_{expL}. Following [19], we know the generalized Hölder's inequality and the following inequalities hold:

$$\frac{1}{|Q|} \int_Q |f(y)g(y)| dy \leqslant ||f||_{\Phi,Q} ||g||_{\tilde{\Phi},Q},$$

$$||f||_{L(logL),Q} \leqslant M_{L(logL)}(f) \leqslant CM^2(f),$$

$$||f - f_{2^jQ}||_{expL,Q} \leqslant Cj||f||_{BMO}.$$

DEFINITION 1. Let φ be a positive, increasing function on R^+ and there exists a constant $D > 0$ such that

$$\varphi(2t) \leqslant D\varphi(t) \ \text{ for } \ t \geqslant 0.$$

Let f be a locally integrable function on R^n. Set, for $1 \leqslant p < \infty$,

$$||f||_{L^{p,\varphi}} = \sup_{x\in R^n,\ d>0} \left(\frac{1}{\varphi(d)} \int_{Q(x,d)} |f(y)|^p dy \right)^{1/p},$$

where $Q(x,d) = \{y \in R^n : |x-y| < d\}$. The generalized Morrey space is defined by

$$L^{p,\varphi}(R^n) = \{f \in L^1_{loc}(R^n) : ||f||_{L^{p,\varphi}} < \infty\}.$$

If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(R^n) = L^{p,\delta}(R^n)$, which is the classical Morrey spaces (see [17], [18]). If $\varphi(d) = 1$, then $L^{p,\varphi}(R^n) = L^p(R^n)$, which is the Lebesgue spaces (see [7]).

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [5], [6], [10], [16]).

In this paper, we will study some singular integral operators as follows (see [1]).

DEFINITION 2. Let $K(x) = \Omega(x)/|x|^n : R^n \setminus \{0\} \to R$. K is said to be a Calderón-Zygmund kernel if

(a) $\Omega \in C^\infty(R^n \setminus \{0\})$;

(b) Ω is homogeneous of degree zero;

(c) $\int_\Sigma \Omega(x)x^\alpha d\sigma(x) = 0$ for all multi-indices $\alpha \in (N \cup \{0\})^n$ with $|\alpha| = N$, where $\Sigma = \{x \in R^n : |x| = 1\}$ is the unit sphere of R^n.

DEFINITION 3. Let $K(x,y) = \Omega(x,y)/|y|^n : R^n \times (R^n \setminus \{0\}) \to R$. K is said to be a variable Calderón-Zygmund kernel if

(d) $K(x,\cdot)$ is a Calderón-Zygmund kernel for a.e. $x \in R^n$;

(e) $\max_{|\gamma|\leqslant 2n} \left|\left| \frac{\partial^{|\gamma|}}{\partial^\gamma y}\Omega(x,y)\right|\right|_{L^\infty(R^n\times\Sigma)} = L < \infty.$

Moreover, let b be a locally integrable function on R^n and T be the singular integral operator with variable Calderón-Zygmund kernel as

$$T(f)(x) = \int_{R^n} K(x, x-y)f(y)dy,$$

where $K(x,x-y) = \frac{\Omega(x,x-y)}{|x-y|^n}$ and that $\Omega(x,y)/|y|^n$ is a variable Calderón-Zygmund kernel. The Toeplitz type operators associated to T is defined by

$$T_b = \sum_{k=1}^m T^{k,1}M_bT^{k,2}$$

and

$$S_b = \sum_{k=1}^m (T^{k,3}M_bI_\alpha T^{k,4} + T^{k,5}I_\alpha M_bT^{k,6}),$$

where $T^{k,1}$ and $T^{k,3}$ are the singular integral operator T with variable Calderón-Zygmund kernel or $\pm I$(the identity operator), $T^{k,2}$, $T^{k,4}$ and $T^{k,6}$ are the bounded linear operators on $L^p(R^n)$ for $1<p<\infty$, $T^{k,5}=\pm I$, $k=1,...,m$, $M_b(f)=bf$ and I_α is the fractional integral operator $(0<\alpha<n)$ (see [2]).

Note that the commutator $[b,T](f)=bT(f)-T(bf)$ is a particular operator of the Toeplitz type operators T_b and S_b. The Toeplitz type operators T_b and S_b are the non-trivial generalizations of the commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [19], [20]). The main purpose of this paper is to prove the M^k-type sharp maximal inequalities for the Toeplitz type operator T_b. As the application, we obtain the boundedness on the Morrey space for the Toeplitz type operators T_b and S_b.

2. Theorems and Lemmas

We shall prove the following theorems.

THEOREM 1. *Let T be the singular integral operator as Definition 3, $0<\eta<1$, $1<s<\infty$ and $b\in BMO(R^n)$. If $T_1(g)=0$ for any $g\in L^r(R^n)$ $(1<r<\infty)$, then there exists a constant $C>0$ such that, for any $f\in C_0^\infty(R^n)$ and $\tilde{x}\in R^n$,*

$$(T_b(f))^\#_\eta(\tilde{x})\leqslant C||b||_{BMO}\sum_{k=1}^m M^2(T^{k,2}(f))(\tilde{x}).$$

THEOREM 2. *Let T be the singular integral operator as Definition 3, $0<\eta<1$, $1<s<\infty$ and $b\in BMO(R^n)$. If $S_1(g)=0$ for any $g\in L^r(R^n)$ $(1<r<\infty)$, then there exists a constant $C>0$ such that, for any $f\in C_0^\infty(R^n)$ and $\tilde{x}\in R^n$,*

$$(S_b(f))^\#_\eta(\tilde{x})\leqslant C||b||_{BMO}\sum_{k=1}^m (M^2(I_\alpha T^{k,4}(f))(\tilde{x})+M_{\alpha,s}(T^{k,6}(f))(\tilde{x})).$$

THEOREM 3. *Let T be the singular integral operator as Definition 3, $1<p<\infty$, $0<D<2^n$ and $b\in BMO(R^n)$. If $T_1(g)=0$ for any $g\in L^r(R^n)$ $(1<r<\infty)$, then T_b is bounded on $L^{p,\varphi}(R^n)$.*

THEOREM 4. *Let T be the singular integral operator as Definition 3, $0<D<2^n$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$ and $b\in BMO(R^n)$. If $S_1(g)=0$ for any $g\in L^r(R^n)$ $(1<r<\infty)$, then S_b is bounded from $L^{p,\varphi}(R^n)$ to $L^{q,\varphi}(R^n)$.*

COROLLARY. *Let $[b,T](f)=bT(f)-T(bf)$ be the commutator generated by the singular integral operator T as Definition 3 and b. Then Theorems 1–4 hold for $[b,T]$.*

To prove the theorems, we need the following lemmas.

LEMMA 1. ([7, p. 485]) *Let $0<p<q<\infty$ and for any function $f\geqslant 0$. We define that, for $1/r=1/p-1/q$,*

$$||f||_{WL^q}=\sup_{\lambda>0}\lambda|\{x\in R^n: f(x)>\lambda\}|^{1/q},\qquad N_{p,q}(f)=\sup_Q||f\chi_Q||_{L^p}/||\chi_Q||_{L^r},$$

where the sup is taken for all measurable sets Q *with* $0<|Q|<\infty$. *Then*

$$||f||_{WL^q} \leqslant N_{p,q}(f) \leqslant (q/(q-p))^{1/p}||f||_{WL^q}.$$

LEMMA 2. (see [1]) *Let* T *be the singular integral operator as Definition 3. Then* T *is bounded on* $L^p(R^n)$ *for* $1<p<\infty$, *and weak* (L^1,L^1) *bounded.*

LEMMA 3. (see [19]) *We have*

$$\frac{1}{|Q|}\int_Q |f(x)g(x)|dx \leqslant ||f||_{expL,Q}||g||_{L(logL),Q}.$$

LEMMA 4. (see [2], [7]) *Let* $0<\alpha<n$, $1\leqslant s<p<n/\alpha$ *and* $1/q=1/p-\alpha/n$. *Then*

$$||M_{\alpha,s}(f)||_{L^q} \leqslant C||f||_{L^p}$$

and

$$||I_\alpha(f)||_{L^q} \leqslant C||f||_{L^p}.$$

LEMMA 5. (see [7]) *Let* $0<p,\eta<\infty$. *Then, for any smooth function* f *for which the left-hand side is finite,*

$$\int_{R^n} M_\eta(f)(x)^p dx \leqslant C\int_{R^n} f_\eta^{\#}(f)(x)^p dx.$$

LEMMA 6. *Let* $0<p,\eta<\infty$ *and* $0<D<2^n$. *Then, for any smooth function* f *for which the left-hand side is finite,*

$$||M_\eta(f)||_{L^{p,\varphi}} \leqslant C||f_\eta^{\#}(f)||_{L^{p,\varphi}}.$$

Proof. For any cube $Q=Q(x_0,d)$ in R^n, we know $M(\chi_Q)\in A_1$ for any cube $Q=Q(x,d)$ by [3]. Noticing that $M(\chi_Q)\leqslant 1$ and $M(\chi_Q)(x)\leqslant d^n/(|x-x_0|-d)^n$ if $x\in Q^c$, by Lemma 5, we have, for $f\in L^{p,\varphi}(R^n)$,

$$\begin{aligned}
&\int_Q M_\eta(f)(x)^p dx = \int_{R^n} M_\eta(f)(x)^p \chi_Q(x)dx\\
&\leqslant \int_{R^n} M_\eta(f)(x)^p M(\chi_Q)(x)dx\\
&\leqslant C\int_{R^n} f_\eta^{\#}(x)^p M(\chi_Q)(x)dx\\
&= C\left(\int_Q f_\eta^{\#}(x)^p M(\chi_Q)(x)dx + \sum_{k=0}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ} f_\eta^{\#}(x)^p M(\chi_Q)(x)dx\right)\\
&\leqslant C\left(\int_Q f_\eta^{\#}(x)^p dx + \sum_{k=0}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ} f_\eta^{\#}(x)^p \frac{|Q|}{|2^{k+1}Q|}dx\right)
\end{aligned}$$

$$\leqslant C\left(\int_Q f_\eta^\#(x)^p dx+\sum_{k=0}^{\infty}\int_{2^{k+1}Q} f_\eta^\#(x)^p 2^{-kn} dy\right)$$

$$\leqslant C||f_\eta^\#||_{L^{p,\varphi}}^p \sum_{k=0}^{\infty} 2^{-kn}\varphi(2^{k+1}d)$$

$$\leqslant C||f_\eta^\#||_{L^{p,\varphi}}^p \sum_{k=0}^{\infty} (2^{-n}D)^k\varphi(d)$$

$$\leqslant C||f_\eta^\#||_{L^{p,\varphi}}^p \varphi(d),$$

thus

$$\left(\frac{1}{\varphi(d)}\int_Q M_\eta(f)(x)^p dx\right)^{1/p} \leqslant C\left(\frac{1}{\varphi(d)}\int_Q f_\eta^\#(x)^p dx\right)^{1/p}$$

and

$$||M_\eta(f)||_{L^{p,\varphi}} \leqslant C||f_\eta^\#||_{L^{p,\varphi}}.$$

This finishes the proof. $\square$

LEMMA 7. *Let $0<\alpha<n$, $0<D<2^n$, $1\leqslant s<p<n/\alpha$ and $1/q=1/p-\alpha/n$. Then*

$$||M_{\alpha,s}(f)||_{L^{q,\varphi}} \leqslant C||f||_{L^{p,\varphi}}$$

and

$$||I_\alpha(f)||_{L^{q,\varphi}} \leqslant C||f||_{L^{p,\varphi}}.$$

LEMMA 8. *Let T be the bounded linear operators on $L^r(R^n)$ for any $1<r<\infty$. Then, for $1<p<\infty$ and $0<D<2^n$,*

$$||T(f)||_{L^{p,\varphi}} \leqslant C||f||_{L^{p,\varphi}}.$$

The proofs of two Lemmas are similar to that of Lemma 6 by Lemma 4, we omit the details.

3. Proofs of Theorems

Proof of Theorem 1. It suffices to prove for $f\in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q |T_b(f)(x)-C_0|^\eta\, dx\right)^{1/\eta} \leqslant C||b||_{BMO}\sum_{k=1}^{m} M^2(T^{k,2}(f))(\tilde{x}).$$

Without loss of generality, we may assume $T^{k,1}$ are T $(k=1,\ldots,m)$. Fix a cube $Q=Q(x_0,d)$ and $\tilde{x}\in Q$. We write, by $T_1(g)=0$,

$$T_b(f)(x)=T_{b-b_{2Q}}(f)(x)=T_{(b-b_{2Q})\chi_{2Q}}(f)(x)+T_{(b-b_{2Q})\chi_{(2Q)^c}}(f)(x)=f_1(x)+f_2(x).$$

Then

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q |T_b(f)(x)-f_2(x_0)|^\eta\,dx\right)^{1/\eta}\\ &\leqslant C\left(\frac{1}{|Q|}\int_Q |f_1(x)|^\eta dx\right)^{1/\eta}+C\left(\frac{1}{|Q|}\int_Q |f_2(x)-f_2(x_0)|^\eta dx\right)^{1/\eta}=I+II.\end{aligned}$$

For I, by Lemma 1, 2 and 3, we obtain

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q |T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(x)|^\eta dx\right)^{1/\eta}\\ &\leqslant |Q|^{-1}\frac{||T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)\chi_Q||_{L^\eta}}{|Q|^{1/\eta-1}}\\ &\leqslant C|Q|^{-1}||T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)||_{WL^1}\\ &\leqslant C|Q|^{-1}||M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)||_{L^1}\\ &\leqslant C|Q|^{-1}\int_{2Q}|b(x)-b_{2Q}||T^{k,2}(f)(x)|dx\\ &\leqslant C||b-b_{2Q}||_{expL,2Q}||T^{k,2}(f)||_{L(logL),2Q}\\ &\leqslant C||b||_{BMO}M^2(T^{k,2}(f))(\tilde{x}),\end{aligned}$$

thus

$$\begin{aligned}I&\leqslant \sum_{k=1}^m\left(\frac{C}{|Q|}\int_Q |T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(x)|^\eta dx\right)^{1/\eta}\\ &\leqslant C||b||_{BMO}\sum_{k=1}^m M^2(T^{k,2}(f))(\tilde{x}).\end{aligned}$$

For II, by [1], we know that

$$T(f)(x)=\sum_{u=1}^\infty\sum_{v=1}^{g_u}a_{uv}(x)\int_{R^n}\frac{Y_{uv}(x-y)}{|x-y|^{n+m}}f(y)dy,$$

where $g_u\leqslant Cu^{n-2}$, $||a_{uv}||_{L^\infty}\leqslant Cu^{-2n}$, $|Y_{uv}(x-y)|\leqslant Cu^{n/2-1}$ and

$$\left|\frac{Y_{uv}(x-y)}{|x-y|^n}-\frac{Y_{uv}(x_0-y)}{|x_0-y|^n}\right|\leqslant Cu^{n/2}|x-x_0|/|x_0-y|^{n+1}$$

for $|x-y|>2|x_0-x|>0$. Then, we get, for $x\in Q$,

$$\begin{aligned}&|T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}}(f)(x)-T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}}T^{k,2}(f)(x_0)|\\ &\leqslant\int_{(2Q)^c}|b(y)-b_{2Q}||K(x,x-y)-K(x_0,x_0-y)||T^{k,2}(f)(y)|dy\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{\infty} \int_{2^j d \leqslant |y-x_0| < 2^{j+1} d} |b(y)-b_{2Q}||K(x,x-y)-K(x_0,x_0-y)||T^{k,2}(f)(y)|dy \\
&\leqslant C \sum_{j=1}^{\infty} \int_{2^j d \leqslant |y-x_0| < 2^{j+1} d} |b(y)-b_{2Q}| \sum_{u=1}^{\infty} \sum_{v=1}^{g_u} |a_{uv}(x)| \\
&\quad \times \left| \frac{Y_{uv}(x-y)}{|x-y|^n} - \frac{Y_{uv}(x_0-y)}{|x_0-y|^n} \right| |T^{k,2}(f)(y)|dy \\
&\leqslant C \sum_{j=1}^{\infty} \int_{2^j d \leqslant |y-x_0| < 2^{j+1} d} |b(y)-b_{2Q}| \frac{|x-x_0|}{|x_0-y|^{n+1}} |T^{k,2}(f)(y)|dy \\
&\leqslant C \sum_{j=1}^{\infty} \frac{d}{(2^{j+1}d)^{n+1}} \int_{2^{j+1}Q} |b(y)-b_{2Q}||T^{k,2}(f)(y)|dy \\
&\leqslant C \sum_{j=1}^{\infty} 2^{-j} \frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} |b(y)-b_{2Q}||T^{k,2}(f)(y)|dy \\
&\leqslant C \sum_{j=1}^{\infty} 2^{-j} ||b-b_{2Q}||_{expL,2^{j+1}Q} ||T^{k,2}(f)||_{L(logL),2^{j+1}Q} \\
&\leqslant C \sum_{j=1}^{\infty} j 2^{-j} ||b||_{BMO} M^2(T^{k,2}(f))(\tilde{x}) \\
&\leqslant C||b||_{BMO} M^2(T^{k,2}(f))(\tilde{x}),
\end{aligned}
$$

thus

$$
\begin{aligned}
II &\leqslant \frac{C}{|Q|} \int_Q \sum_{k=1}^{m} |T^{k,1} M_{(b-b_{2Q})\chi_{(2Q)^c}} T^{k,2}(f)(x) - T^{k,1} M_{(b-b_{2Q})\chi_{(2Q)^c}} T^{k,2}(f)(x_0)|dx \\
&\leqslant C||b||_{BMO} \sum_{k=1}^{m} M^2(T^{k,2}(f))(\tilde{x}).
\end{aligned}
$$

This completes the proof of Theorem 1. $\square$

Proof of Theorem 2. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$
\left(\frac{1}{|Q|} \int_Q |S_b(f)(x) - C_0|^\eta \, dx \right)^{1/\eta} \leqslant C||b||_{BMO} \sum_{k=1}^{m} (M^2(I_\alpha T^{k,4}(f))(\tilde{x}) + M_{\alpha,s}(T^{k,6}(f))(\tilde{x})).
$$

Without loss of generality, we may assume $T^{k,3}$ are $T (k = 1,...,m)$. Fix a cube $Q = Q(x_0,d)$ and $\tilde{x} \in Q$. Write, by $T_1(g) = 0$,

$$
\begin{aligned}
S_b(f)(x) &= \sum_{k=1}^{m} T^{k,3} M_b I_\alpha T^{k,4}(f)(x) + \sum_{k=1}^{m} T^{k,5} I_\alpha M_b T^{k,6}(f)(x) \\
&= A_b(x) + B_b(x) = A_{b-b_Q}(x) + B_{b-b_Q}(x),
\end{aligned}
$$

where

$$\begin{aligned} A_{b-b_Q}(x) &= \sum_{k=1}^m T^{k,3} M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{k,4}(f)(x) + \sum_{k=1}^m T^{k,3} M_{(b-b_Q)\chi_{(2Q)^c}} I_\alpha T^{k,4}(f)(x) \\ &= A_1(x) + A_2(x) \end{aligned}$$

and

$$\begin{aligned} B_{b-b_Q}(x) &= \sum_{k=1}^m T^{k,5} I_\alpha M_{(b-b_Q)\chi_{2Q}} T^{k,6}(f)(x) + \sum_{k=1}^m T^{k,5} I_\alpha M_{(b-b_Q)\chi_{(2Q)^c}} T^{k,6}(f)(x) \\ &= B_1(x) + B_2(x). \end{aligned}$$

Then

$$\begin{aligned} &\left(\frac{1}{|Q|}\int_Q \left|A_{b-b_Q}(f)(x) - A_2(x_0)\right|^\eta dx\right)^{1/\eta} \\ &\leqslant C\left(\frac{1}{|Q|}\int_Q |A_1(x)|^\eta dx\right)^{1/\eta} + C\left(\frac{1}{|Q|}\int_Q |A_2(x) - A_2(x_0)|^\eta dx\right)^{1/\eta} \\ &= I_1 + I_2 \end{aligned}$$

and

$$\begin{aligned} &\left(\frac{1}{|Q|}\int_Q \left|B_{b-b_Q}(f)(x) - B_2(x_0)\right|^\eta dx\right)^{1/\eta} \\ &\leqslant C\left(\frac{1}{|Q|}\int_Q |B_1(x)|^\eta dx\right)^{1/\eta} + C\left(\frac{1}{|Q|}\int_Q |B_2(x) - B_2(x_0)|^\eta dx\right)^{1/\eta} \\ &= I_3 + I_4. \end{aligned}$$

For I_1 and I_2, by using the same argument as in the proof of Theorem 1, we get

$$\begin{aligned} I_1 &\leqslant C\sum_{k=1}^m \left(\frac{1}{|Q|}\int_{R^n} |T^{k,3} M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{k,4}(f)(x)|^\eta dx\right)^{1/\eta} \\ &\leqslant \sum_{k=1}^m |Q|^{-1} \frac{||T^{k,3} M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{k,4}(f)\chi_Q||_{L^\eta}}{|Q|^{1/\eta-1}} \\ &\leqslant C\sum_{k=1}^m |Q|^{-1} ||T^{k,3} M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{k,4}(f)||_{WL^1} \\ &\leqslant C\sum_{k=1}^m |Q|^{-1} ||M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{k,4}(f)||_{L^1} \\ &\leqslant C\sum_{k=1}^m |Q|^{-1} \int_{2Q} |b(x) - b_{2Q}||I_\alpha T^{k,4}(f)(x)| dx \\ &\leqslant C\sum_{k=1}^m ||b - b_{2Q}||_{expL,2Q} ||I_\alpha T^{k,4}(f)||_{L(logL),2Q} \\ &\leqslant C||b||_{BMO} \sum_{k=1}^m M^2(I_\alpha T^{k,4}(f))(\tilde{x}), \end{aligned}$$

$$
\begin{aligned}
I_2 &\leqslant C\sum_{k=1}^{m}\frac{1}{|Q|}\int_Q\sum_{j=1}^{\infty}\int_{2^jd\leqslant|y-x_0|<2^{j+1}d}|b(y)-b_{2Q}||K(x,x-y)-K(x_0,x_0-y)|\\
&\quad\times|I_\alpha T^{k,4}(f)(y)|dydx\\
&\leqslant C\sum_{k=1}^{m}\frac{1}{|Q|}\int_Q\sum_{j=1}^{\infty}\int_{2^jd\leqslant|y-x_0|<2^{j+1}d}|b(y)-b_{2Q}|\frac{|x-x_0|}{|x_0-y|^{n+1}}|I_\alpha T^{k,4}(f)(y)|dydx\\
&\leqslant C\sum_{k=1}^{m}\sum_{j=1}^{\infty}2^{-j}\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|b(y)-b_{2Q}||I_\alpha T^{k,4}(f)(y)|dy\\
&\leqslant C\sum_{k=1}^{m}\sum_{j=1}^{\infty}2^{-j}||b-b_{2Q}||_{expL,2^{j+1}Q}||I_\alpha T^{k,4}(f)||_{L(logL),2^{j+1}Q}\\
&\leqslant C\sum_{k=1}^{m}\sum_{j=1}^{\infty}j2^{-j}||b||_{BMO}M^2(I_\alpha T^{k,4}(f))(\tilde{x})\\
&\leqslant C||b||_{BMO}\sum_{k=1}^{m}M^2(I_\alpha T^{k,4}(f))(\tilde{x}).
\end{aligned}
$$

For I_3, by Lemma 4, we get, for $1<p<\min(s,n/\alpha)$ and $1/q=1/p-\alpha/n$,

$$
\begin{aligned}
I_3 &\leqslant C\sum_{k=1}^{m}\left(\frac{1}{|Q|}\int_{R^n}|I_\alpha M_{(b-b_Q)\chi_{2Q}}T^{k,6}(f)(x)|^q dx\right)^{1/q}\\
&\leqslant C\sum_{k=1}^{m}|Q|^{-1/q}\left[\int_{2Q}(|b(x)-b_Q||T^{k,6}(f)(x)|)^p dx\right]^{1/p}\\
&\leqslant C\sum_{k=1}^{m}\left(\frac{1}{|2Q|}\int_{2Q}|b(x)-b_Q|^{ps/(s-p)}dx\right)^{(s-p)/ps}\left(\frac{1}{|2Q|^{1-s\alpha/n}}\int_{2Q}|T^{k,6}(f)(x)|^s dx\right)^{1/s}\\
&\leqslant C||b||_{BMO}\sum_{k=1}^{m}M_{\alpha,s}(T^{k,6}(f))(\tilde{x}).
\end{aligned}
$$

For I_4, we get

$$
\begin{aligned}
I_4 &\leqslant |Q|^{-1}\sum_{k=1}^{m}\int_Q\int_{(2Q)^c}|b(y)-b_{2Q}|\left|\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|x_0-y|^{n-\alpha}}\right||T^{k,6}(f)(y)|dydx\\
&\leqslant C\sum_{k=1}^{m}\sum_{j=1}^{\infty}\int_{2^jd\leqslant|y-x_0|<2^{j+1}d}|b(y)-b_{2Q}|\frac{d}{|x_0-y|^{n-\alpha+1}}|T^{k,6}(f)(y)|dy\\
&\leqslant C\sum_{k=1}^{m}\sum_{j=1}^{\infty}d(2^jd)^{-n+\alpha-1}(2^jd)^{n(1-1/s)}(2^jd)^{n/s-\alpha}\\
&\quad\times\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|b(y)-b_Q|^{s'}dy\right)^{1/s'}\left(\frac{1}{|2^{j+1}Q|^{1-s\alpha/n}}\int_{2^{j+1}Q}|T^{k,6}(f)(y)|^s dy\right)^{1/s}
\end{aligned}
$$

$$\leqslant C||b||_{BMO}\sum_{k=1}^{m} M_{\alpha,s}(T^{k,6}(f))(\tilde{x})\sum_{j=1}^{\infty} j2^{-j}$$
$$\leqslant C||b||_{BMO}\sum_{k=1}^{m} M_{\alpha,s}(T^{k,6}(f))(\tilde{x}).$$

This completes the proof of Theorem 2. □

Proof of Theorem 3. By Theorem 1 and Lemmas 7–8, we have

$$\begin{aligned}
||T_b(f)||_{L^{p,\varphi}} &\leqslant \|M_\eta(T_b(f))\|_{L^{p,\varphi}} \leqslant C\|(T_b(f))^{\#}_{\eta}\|_{L^{p,\varphi}}\\
&\leqslant C||b||_{BMO}\sum_{k=1}^{m}\|M^2(T^{k,2}(f))\|_{L^{p,\varphi}}\\
&\leqslant C||b||_{BMO}\sum_{k=1}^{m}\|T^{k,2}(f)\|_{L^{p,\varphi}}\\
&\leqslant C||b||_{BMO}\|f\|_{L^{p,\varphi}}.
\end{aligned}$$

This completes the proof of Theorem 3. □

Proof of Theorem 4. Choose $1 < s < p$ in Theorem 2, then we have, by Lemmas 7 and 8,

$$\begin{aligned}
||S_b(f)||_{L^{q,\varphi}} &\leqslant \|M_\eta(S_b(f))\|_{L^{q,\varphi}} \leqslant C\|(S_b(f))^{\#}_{\eta}\|_{L^{q,\varphi}}\\
&\leqslant C||b||_{BMO}\sum_{k=1}^{m}(\|M^2(I_\alpha T^{k,4}(f))\|_{L^{q,\varphi}} + \|M_{\alpha,s}(T^{k,6}(f))\|_{L^{q,\varphi}})\\
&\leqslant C||b||_{BMO}\sum_{k=1}^{m}(\|I_\alpha T^{k,4}(f)\|_{L^{q,\varphi}} + \|T^{k,6}(f)\|_{L^{p,\varphi}})\\
&\leqslant C||b||_{BMO}\sum_{k=1}^{m}(\|T^{k,4}(f)\|_{L^{p,\varphi}} + \|f\|_{L^{p,\varphi}})\\
&\leqslant C||b||_{BMO}\|f\|_{L^{p,\varphi}}.
\end{aligned}$$

This completes the proof of Theorem 4. □

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