BOUNDEDNESS ON MORREY SPACE FOR TOEPLITZ TYPE OPERATOR ASSOCIATED TO SINGULAR INTEGRAL OPERATOR WITH VARIABLE CALDERÓN–ZYGMUND KERNEL

CHUANGXIA HUANG, SHENG GUO AND LANZHE LIU

(Communicated by N. Elezović)

Abstract. In this paper, the boundedness of the Toeplitz type operators associated to the singular integral operator with variable Calderón-Zygmund kernel on Morrey spaces is obtained. For this purpose, some \( M^k \)-type sharp maximal function inequalities for the operators are proved.

1. Introduction and Preliminaries

As the development of singular integral operators (see [7], [21]), their commutators have been well studied. In [4], [19], [20], the authors proved that the commutators generated by the singular integral operators and \( BMO \) functions are bounded on \( L^p(\mathbb{R}^n) \) for \( 1 < p < \infty \). Chanillo (see [2]) proved a similar result when singular integral operators are replaced by the fractional integral operators. In [1], Calderón and Zygmund introduced some singular integral operators with variable kernel and discussed their boundedness. In [11–13], [22], the authors obtained the boundedness for the commutators generated by the singular integral operators with variable kernel and \( BMO \) functions. In [15], the authors proved the boundedness for the multilinear oscillatory singular integral operators generated by the operators and \( BMO \) functions. In [8], [9], [14], some Toeplitz type operators associated to the singular integral operators and strongly singular integral operators are introduced, and the boundedness for the operators generated by \( BMO \) and Lipschitz functions is obtained. Motivated by these, in this paper, we will study the Toeplitz type operator generated by the singular integral operator with variable Calderón-Zygmund kernel.

First, let us introduce some notations. Throughout this paper, \( Q \) will denote a cube of \( \mathbb{R}^n \) with sides parallel to the axes. For any locally integrable function \( f \), the sharp maximal function of \( f \) is defined by

\[
 f^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy,
\]
where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)dx$. It is well-known that (see [7], [21])

$$f^\#(x) \approx \sup_{Q:x \in C} \frac{1}{|Q|} \int_Q |f(y) - c|dy.$$  

We say that $f$ belongs to $BMO(\mathbb{R}^n)$ if $f^\#$ belongs to $L^\infty(\mathbb{R}^n)$ and define $\|f\|_{BMO} = \|f^\#\|_{L^\infty}$. It is known that (see [21])

$$\|f - f_{2kQ}\|_{BMO} \leq Ck \|f\|_{BMO}.$$  

For $0 < r < \infty$, we denote $f_r^\#$ by

$$f_r^\#(x) = [(|f|^r)^\#(x)]^{1/r}.$$  

Let

$$M(f)(x) = \sup_{Q:x \in C} \frac{1}{|Q|} \int_Q |f(y)|dy.$$  

For $\eta > 0$, let $M_\eta(f) = M((|f|^{\eta})^{1/\eta})$. For $k \in \mathbb{N}$, we denote by $M^k$ the operator $M$ iterated $k$ times, i.e., $M^1(f) = M(f)$ and

$$M^k(f) = M(M^{k-1}(f)) \text{ when } k \geq 2.$$  

For $0 < \eta < n$ and $1 \leq r < \infty$, set

$$M_{\eta,r}(f)(x) = \sup_{Q:x \in C} \left( \frac{1}{|Q|^{1-r\eta/n}} \int_Q |f(y)|^rdy \right)^{1/r}.$$  

Let $\Phi$ be a Young function and $\tilde{\Phi}$ be the complementary associated to $\Phi$, we denote the $\Phi$-average for a function $f$ by

$$\|f\|_{\Phi,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi \left( \frac{|f(y)|}{\lambda} \right) dy \leq 1 \right\}$$  

and the maximal function associated to $\Phi$ by

$$M_{\Phi}(f)(x) = \sup_{x \in Q} \|f\|_{\Phi,Q}.$$  

The Young functions which we use in this paper are $\Phi(t) = t(1 + \log t)$ and $\tilde{\Phi}(t) = \exp(t)$, the corresponding average and maximal functions are denoted by $\|\cdot\|_{L(\log L),Q}$, $M_{L(\log L)}$ and $\|\cdot\|_{\exp L, Q}$, $M_{\exp L}$. Following [19], we know the generalized H"{o}lder’s inequality and the following inequalities hold:

$$\frac{1}{|Q|} \int_Q |f(y)g(y)|dy \leq \|f\|_{\Phi,Q} \|g\|_{\Phi,Q},$$  

$$\|f\|_{L(\log L), Q} \leq M_{L(\log L)}(f) \leq CM^2(f),$$

where $f_Q = |Q|^{-1} \int_Q f(x)dx$. It is well-known that (see [7], [21])

$$f^\#(x) \approx \sup_{Q:x \in C} \frac{1}{|Q|} \int_Q |f(y) - c|dy.$$  

We say that $f$ belongs to $BMO(\mathbb{R}^n)$ if $f^\#$ belongs to $L^\infty(\mathbb{R}^n)$ and define $\|f\|_{BMO} = \|f^\#\|_{L^\infty}$. It is known that (see [21])

$$\|f - f_{2kQ}\|_{BMO} \leq Ck \|f\|_{BMO}.$$  

For $0 < r < \infty$, we denote $f_r^\#$ by

$$f_r^\#(x) = [(|f|^r)^\#(x)]^{1/r}.$$  

Let

$$M(f)(x) = \sup_{Q:x \in C} \frac{1}{|Q|} \int_Q |f(y)|dy.$$  

For $\eta > 0$, let $M_\eta(f) = M((|f|^{\eta})^{1/\eta})$. For $k \in \mathbb{N}$, we denote by $M^k$ the operator $M$ iterated $k$ times, i.e., $M^1(f) = M(f)$ and

$$M^k(f) = M(M^{k-1}(f)) \text{ when } k \geq 2.$$  

For $0 < \eta < n$ and $1 \leq r < \infty$, set

$$M_{\eta,r}(f)(x) = \sup_{Q:x \in C} \left( \frac{1}{|Q|^{1-r\eta/n}} \int_Q |f(y)|^rdy \right)^{1/r}.$$  

Let $\Phi$ be a Young function and $\tilde{\Phi}$ be the complementary associated to $\Phi$, we denote the $\Phi$-average for a function $f$ by

$$\|f\|_{\Phi,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi \left( \frac{|f(y)|}{\lambda} \right) dy \leq 1 \right\}$$  

and the maximal function associated to $\Phi$ by

$$M_{\Phi}(f)(x) = \sup_{x \in Q} \|f\|_{\Phi,Q}.$$  

The Young functions which we use in this paper are $\Phi(t) = t(1 + \log t)$ and $\tilde{\Phi}(t) = \exp(t)$, the corresponding average and maximal functions are denoted by $\|\cdot\|_{L(\log L),Q}$, $M_{L(\log L)}$ and $\|\cdot\|_{\exp L, Q}$, $M_{\exp L}$. Following [19], we know the generalized H"{o}lder’s inequality and the following inequalities hold:

$$\frac{1}{|Q|} \int_Q |f(y)g(y)|dy \leq \|f\|_{\Phi,Q} \|g\|_{\Phi,Q},$$  

$$\|f\|_{L(\log L), Q} \leq M_{L(\log L)}(f) \leq CM^2(f).$$
\[ \left\| f - f_{2L} \right\|_{\exp L} \leq Cj \left\| f \right\|_{\text{BMO}}. \]

**Definition 1.** Let \( \varphi \) be a positive, increasing function on \( R^+ \) and there exists a constant \( D > 0 \) such that
\[ \varphi(2t) \leq D\varphi(t) \quad \text{for} \quad t \geq 0. \]
Let \( f \) be a locally integrable function on \( R^n \). Set, for \( 1 \leq p < \infty \),
\[ \left\| f \right\|_{L^p, \varphi} = \sup_{x \in \Omega, \ d > 0} \left( \frac{1}{\varphi(d)} \int_{Q(x, d)} |f(y)|^p \, dy \right)^{1/p}, \]
where \( Q(x, d) = \{ y \in R^n : |x - y| < d \} \). The generalized Morrey space is defined by
\[ L^{p, \varphi}(R^n) = \{ f \in L^1_{\text{loc}}(R^n) : \left\| f \right\|_{L^p, \varphi} < \infty \} . \]

If \( \varphi(d) = d^\delta, \delta > 0 \), then \( L^{p, \varphi}(R^n) = L^{p, \delta}(R^n) \), which is the classical Morrey spaces (see [17], [18]). If \( \varphi(d) = 1 \), then \( L^{p, \varphi}(R^n) = L^p(R^n) \), which is the Lebesgue spaces (see [7]).

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [5], [6], [10], [16]).

In this paper, we will study some singular integral operators as follows (see [1]).

**Definition 2.** Let \( K(x) = \Omega(x)/|x|^n : R^n \setminus \{0\} \to R \). \( K \) is said to be a Calderón-Zygmund kernel if
(a) \( \Omega \in \mathcal{C}^\infty(R^n \setminus \{0\}) \);
(b) \( \Omega \) is homogeneous of degree zero;
(c) \( \int_\Sigma \Omega(x) x^\alpha \, d\sigma(x) = 0 \) for all multi-indices \( \alpha \in (\mathbb{N} \cup \{0\})^n \) with \( |\alpha| = N \), where \( \Sigma = \{ x \in R^n : |x| = 1 \} \) is the unit sphere of \( R^n \).

**Definition 3.** Let \( K(x, y) = \Omega(x, y)/|y|^n : R^n \times (R^n \setminus \{0\}) \to R \). \( K \) is said to be a variable Calderón-Zygmund kernel if
(d) \( K(x, \cdot) \) is a Calderón-Zygmund kernel for a.e. \( x \in R^n \);
(e) \( \max_{|y| \leq 2n} \left\| \frac{\partial^{h_y} \Omega(x, y)}{\partial y} \right\|_{L^\infty(R^n \setminus \Sigma)} = L < \infty \).

Moreover, let \( b \) be a locally integrable function on \( R^n \) and \( T \) be the singular integral operator with variable Calderón-Zygmund kernel as
\[ T(f)(x) = \int_{R^n} K(x, x - y) f(y) \, dy, \]
where \( K(x, x - y) = \frac{\Omega(x, x - y)}{|x - y|^n} \) and that \( \Omega(x, y)/|y|^n \) is a variable Calderón-Zygmund kernel. The Toeplitz type operators associated to \( T \) is defined by
\[ T_b = \sum_{k=1}^m T^{k, 1} M_b T^{k, 2}, \]
and
\[ S_b = \sum_{k=1}^m (T^{k, 3} M_b I_\alpha T^{k, 4} + T^{k, 5} I_\alpha M_b T^{k, 6}), \]
where $T^{k,1}$ and $T^{k,3}$ are the singular integral operator $T$ with variable Calderón-Zygmund kernel or $\pm I$ (the identity operator), $T^{k,2}$, $T^{k,4}$ and $T^{k,6}$ are the bounded linear operators on $L^p(R^n)$ for $1 < p < \infty$, $T^{k,5} = \pm I$, $k = 1, ..., m$, $M_b(f) = bf$ and $I_\alpha$ is the fractional integral operator $(0 < \alpha < n)$ (see [2]).

Note that the commutator $[b, T](f) = bT(f) - T(bf)$ is a particular operator of the Toeplitz type operators $T_b$ and $S_b$. The Toeplitz type operators $T_b$ and $S_b$ are the non-trivial generalizations of the commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [19], [20]). The main purpose of this paper is to prove the $M^k$-type sharp maximal inequalities for the Toeplitz type operator $T_b$. As the application, we obtain the boundedness on the Morrey space for the Toeplitz type operators $T_b$ and $S_b$.

2. Theorems and Lemmas

We shall prove the following theorems.

**Theorem 1.** Let $T$ be the singular integral operator as Definition 3, $0 < \eta < 1$, $1 < s < \infty$ and $b \in \text{BMO}(R^n)$. If $T_1(g) = 0$ for any $g \in L^s(R^n)$ $(1 < r < \infty)$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $x \in R^n$,

$$\left( T_b(f) \right)_\eta^\#(x) \leq C ||b||_{\text{BMO}} \sum_{k=1}^{m} M^2(T^{k,2}(f))(x).$$

**Theorem 2.** Let $T$ be the singular integral operator as Definition 3, $0 < \eta < 1$, $1 < s < \infty$ and $b \in \text{BMO}(R^n)$. If $S_1(g) = 0$ for any $g \in L^s(R^n)$ $(1 < r < \infty)$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $x \in R^n$,

$$\left( S_b(f) \right)_\eta^\#(x) \leq C ||b||_{\text{BMO}} \sum_{k=1}^{m} (M^2(I_\alpha T^{k,4}(f))(x) + M_{\alpha,s}(T^{k,6}(f))(x)).$$

**Theorem 3.** Let $T$ be the singular integral operator as Definition 3, $1 < p < \infty$, $0 < D < 2^n$ and $b \in \text{BMO}(R^n)$. If $T_1(g) = 0$ for any $g \in L^p(R^n)$ $(1 < r < \infty)$, then $T_b$ is bounded on $L^{p,\Phi}(R^n)$.

**Theorem 4.** Let $T$ be the singular integral operator as Definition 3, $0 < D < 2^n$, $1 < p < n/\alpha$. $1/q = 1/p - \alpha/n$ and $b \in \text{BMO}(R^n)$. If $S_1(g) = 0$ for any $g \in L^p(R^n)$ $(1 < r < \infty)$, then $S_b$ is bounded from $L^{p,\Phi}(R^n)$ to $L^{q,\Phi}(R^n)$.

**Corollary.** Let $[b, T](f) = bT(f) - T(bf)$ be the commutator generated by the singular integral operator $T$ as Definition 3 and $b$. Then Theorems 1–4 hold for $[b, T]$.

To prove the theorems, we need the following lemmas.

**Lemma 1.** ([7, p. 485]) Let $0 < p < q < \infty$ and for any function $f \geq 0$. We define that, for $1/r = 1/p - 1/q$,

$$\|f\|_{WL^q} = \sup_{\lambda > 0} \lambda \|\{x \in R^n : f(x) > \lambda\}\|^{1/q}, \quad N_{p,q}(f) = \sup_Q \|f \chi_Q\|_{L^p}/\|\chi_Q\|_{L^r},$$
where the sup is taken for all measurable sets $Q$ with $0 < |Q| < \infty$. Then
\[
\|f\|_{W^{L_q}} \leq N_{p,q}(f) \leq (q/(q-p))^{1/p}\|f\|_{W^{L_q}}.
\]

**Lemma 2.** (see [1]) Let $T$ be the singular integral operator as Definition 3. Then $T$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, and weak $(L^1, L^1)$ bounded.

**Lemma 3.** (see [19]) We have
\[
\frac{1}{|Q|} \int_Q |f(x)g(x)| dx \leq \|f\|_{\exp L^p} \|g\|_{L^{(\log L)^{\frac{1}{p}}}}.
\]

**Lemma 4.** (see [2], [7]) Let $0 < \alpha < n$, $1 \leq s < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Then
\[
\|M_{\alpha,s}(f)\|_{L^q} \leq C\|f\|_{L^p}
\]
and
\[
\|I_\alpha(f)\|_{L^q} \leq C\|f\|_{L^p}.
\]

**Lemma 5.** (see [7]) Let $0 < p, \eta < \infty$. Then, for any smooth function $f$ for which the left-hand side is finite,
\[
\int_{\mathbb{R}^n} M_\eta(f)(x)^p dx \leq C \int_{\mathbb{R}^n} f_\eta^p(x)^p dx.
\]

**Lemma 6.** Let $0 < p, \eta < \infty$ and $0 < D < 2^n$. Then, for any smooth function $f$ for which the left-hand side is finite,
\[
\|M_\eta(f)\|_{L^p,\phi} \leq C\|f_\eta^p(f)\|_{L^p,\phi}.
\]

**Proof.** For any cube $Q = Q(x_0, d)$ in $\mathbb{R}^n$, we know $M(\chi_Q) \in A_1$ for any cube $Q = Q(x, d)$ by [3]. Noticing that $M(\chi_Q) \leq 1$ and $M(\chi_Q)(x) \leq d^n/(|x - x_0| - d)^n$ if $x \in Q^*$, by Lemma 5, we have, for $f \in L^{p,\phi}(\mathbb{R}^n)$,
\[
\int_{\mathbb{R}^n} M_\eta(f)(x)^p dx = \int_{\mathbb{R}^n} M_\eta(f)(x)^p \chi_Q(x) dx
\]
\[
\leq \int_{\mathbb{R}^n} M_\eta(f)(x)^p M(\chi_Q)(x) dx
\]
\[
\leq C \int_{\mathbb{R}^n} f_\eta^p(x)^p M(\chi_Q)(x) dx
\]
\[
= C \left( \int_Q f_\eta^p(x)^p M(\chi_Q)(x) dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^kQ} f_\eta^p(x)^p M(\chi_Q)(x) dx \right)
\]
\[
\leq C \left( \int_Q f_\eta^p(x)^p dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^kQ} f_\eta^p(x)^p \frac{|Q|}{|2^{k+1}Q|} dx \right)
\]
\[ C \left( \int_Q f_\eta^#(x)^p \, dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q} f_\eta^#(x)^p 2^{-kn} \, dy \right) \]
\[ \leq C \left( f_\eta^# \right)_L \sum_{k=0}^{\infty} 2^{-kn} \varphi(2^{k+1}d) \]
\[ \leq C \left( f_\eta^# \right)_L \sum_{k=0}^{\infty} (2^{-n}D)^k \varphi(d) \]
\[ \leq C \left( f_\eta^# \right)_L \varphi(d), \]
thus
\[ \left( \frac{1}{\varphi(d)} \int_Q M_\eta(f)(x)^p \, dx \right)^{1/p} \leq C \left( \frac{1}{\varphi(d)} \int_Q f_\eta^#(x)^p \, dx \right)^{1/p} \]
and
\[ \|M_\eta(f)\|_{L^p,\varphi} \leq C \|f_\eta^#\|_{L^p,\varphi}. \]

This finishes the proof. \( \square \)

**Lemma 7.** Let \( 0 < \alpha < n, 0 < D < 2^n, 1 \leq s < p < n/\alpha \) and \( 1/q = 1/p - \alpha/n \). Then
\[ \|M_{\alpha,s}(f)\|_{L^q,\varphi} \leq C \|f\|_{L^p,\varphi} \]
and
\[ \|I_\alpha(f)\|_{L^q,\varphi} \leq C \|f\|_{L^p,\varphi}. \]

**Lemma 8.** Let \( T \) be the bounded linear operators on \( L^r(R^n) \) for any \( 1 < r < \infty \). Then, for \( 1 < p < \infty \) and \( 0 < D < 2^n \),
\[ \|T(f)\|_{L^p,\varphi} \leq C \|f\|_{L^p,\varphi}. \]

The proofs of two Lemmas are similar to that of Lemma 6 by Lemma 4, we omit the details.

3. Proofs of Theorems

**Proof of Theorem 1.** It suffices to prove for \( f \in C^\infty_0(R^n) \) and some constant \( C_0 \), the following inequality holds:
\[ \left( \frac{1}{|Q|} \int_Q |T_b(f)(x) - C_0|^\eta \, dx \right)^{1/\eta} \leq C \|b\|_{BMO} \sum_{k=1}^{m} M^2(T^{k,2}(f))(\bar{x}). \]
Without loss of generality, we may assume \( T^{k,1} \) are \( T \ (k = 1, \ldots, m) \). Fix a cube \( Q = Q(x_0,d) \) and \( \bar{x} \in Q \). We write, by \( T_1(g) = 0 \),
\[ T_b(f)(x) = T_{b-b_{2Q}}(f)(x) = T_{(b-b_{2Q})\chi_{2Q}}(f)(x) + T_{(b-b_{2Q})\chi_{(2Q)^c}}(f)(x) = f_1(x) + f_2(x). \]
Then
\[
\left( \frac{1}{|Q|} \int_Q |T_b(f)(x) - f_2(x_0)|^\eta \, dx \right)^{1/\eta} 
\leq C \left( \frac{1}{|Q|} \int_Q |f_1(x)|^\eta \, dx \right)^{1/\eta} + C \left( \frac{1}{|Q|} \int_Q |f_2(x) - f_2(x_0)|^\eta \, dx \right)^{1/\eta} = I + II.
\]

For $I$, by Lemma 1, 2 and 3, we obtain
\[
\left( \frac{1}{|Q|} \int_Q |T^{k,1} M_{(b-b_2)Q} T^{k,2}(f)(x)|^\eta \, dx \right)^{1/\eta}
\leq |Q|^{-1} \left\| T^{k,1} M_{(b-b_2)Q} T^{k,2}(f) \right\|_{L^\eta}
\leq C|Q|^{-1} \left\| T^{k,1} M_{(b-b_2)Q} T^{k,2}(f) \right\|_{WL^1}
\leq C|Q|^{-1} \left\| M_{(b-b_2)Q} T^{k,2}(f) \right\|_{L^1}
\leq C|Q|^{-1} \int_{2Q} |b(x) - b_2Q| T^{k,2}(f)(x) |dx
\leq C||b - b_2Q||_{expL,2Q} \left\| T^{k,2}(f) \right\|_{L^2(\log L),2Q}
\leq C||b||_{BMO} M^2(T^{k,2}(f))(\bar{x}),
\]
thus
\[
I \leq \sum_{k=1}^m \left( \frac{C}{|Q|} \int_Q |T^{k,1} M_{(b-b_2)Q} T^{k,2}(f)(x)|^\eta \, dx \right)^{1/\eta}
\leq C||b||_{BMO} \sum_{k=1}^m M^2(T^{k,2}(f))(\bar{x}).
\]

For $II$, by [1], we know that
\[
T(f)(x) = \sum_{u=1}^\infty \sum_{v=1}^{g_u} a_{uv}(x) \int_{\mathbb{R}^n} \frac{Y_{uv}(x-y)}{|x-y|^{n+m}} f(y) \, dy,
\]
where $g_u \leq Cu^{n-2}$, $||a_{uv}||_{L^\infty} \leq Cu^{-2n}$, $|Y_{uv}(x-y)| \leq Cu^{n/2-1}$ and
\[
\left| \frac{Y_{uv}(x-y)}{|x-y|^n} - \frac{Y_{uv}(x_0-y)}{|x_0-y|^n} \right| \leq Cu^{n/2} |x-x_0|/|x_0-y|^{n+1}
\]
for $|x-y| > 2|x_0-x| > 0$. Then, we get, for $x \in Q$,
\[
\left| T^{k,1} M_{(b-b_2)Q} T^{k,2}(f)(x) - T^{k,1} M_{(b-b_2)Q} T^{k,2}(f)(x_0) \right|
\leq \int_{(2Q)^c} |b(y) - b_2Q| K(x,x-y) - K(x_0,x_0-y) ||T^{k,2}(f)(y)| \, dy
\]
\[ = \sum_{j=1}^{\infty} \int_{2^{-j}d \leq |y - x_0| < 2^{j+1}d} |b(y) - b_{2Q}| |K(x, x - y) - K(x_0, x_0 - y)||T^{k,2}(f)(y)| dy \]
\[ \leq C \sum_{j=1}^{\infty} \int_{2^{-j}d \leq |y - x_0| < 2^{j+1}d} |b(y) - b_{2Q}| \sum_{u=1}^{8} \sum_{v=1}^{8} |a_{uv}(x)| \]
\[ \times \frac{|Y_{uv}(x - y)|}{|x - y|^n} \cdot \frac{|Y_{uv}(x_0 - y)|}{|x_0 - y|^n} \cdot |T^{k,2}(f)(y)| dy \]
\[ \leq C \sum_{j=1}^{\infty} \int_{2^{-j}d \leq |y - x_0| < 2^{j+1}d} |b(y) - b_{2Q}| \frac{|x - x_0|}{|x_0 - y|^n+1} |T^{k,2}(f)(y)| dy \]
\[ \leq C \sum_{j=1}^{\infty} \frac{d}{(2^{j+1}d)^{n+1}} \int_{2^{j+1}Q} |b(y) - b_{2Q}| |T^{k,2}(f)(y)| dy \]
\[ \leq C \sum_{j=1}^{\infty} 2^{-j} \frac{1}{2^{j+1}Q} \int_{2^{j+1}Q} |b(y) - b_{2Q}| |T^{k,2}(f)(y)| dy \]
\[ \leq C \sum_{j=1}^{\infty} 2^{-j} |b - b_{2Q}|_{\text{exp}, 2^{j+1}Q} |T^{k,2}(f)|_{L(\log L), 2^{j+1}Q} \]
\[ \leq C \sum_{j=1}^{\infty} j 2^{-j} |b|_{BMO} M^2(T^{k,2}(f))(\bar{x}) \]
\[ \leq C |b|_{BMO} M^2(T^{k,2}(f))(\bar{x}), \]

thus
\[ II \leq C \frac{1}{|Q|} \int_{Q} \sum_{k=1}^{m} |T^{k,1}M_{b-b_{2Q}}\chi_{(\not\in Q)} \cdot T^{k,2}(f)(x) - T^{k,1}M_{b-b_{2Q}}\chi_{(\not\in Q)} \cdot T^{k,2}(f)(x_0)| dx \]
\[ \leq C |b|_{BMO} \sum_{k=1}^{m} M^2(T^{k,2}(f))(\bar{x}). \]

This completes the proof of Theorem 1. \(\square\)

**Proof of Theorem 2.** It suffices to prove for \(f \in C_0^\infty(R^n)\) and some constant \(C_0\), the following inequality holds:
\[ \left( \frac{1}{|Q|} \int_{Q} |S_b(f)(x) - C_0|^{\eta} dx \right)^{1/\eta} \leq C |b|_{BMO} \sum_{k=1}^{m} (M^2(J^{k,4}(f))(\bar{x}) + M_{\alpha,5}(T^{k,6}(f))(\bar{x})). \]
Without loss of generality, we may assume \(T^{k,3}\) are \(T(k = 1, \ldots, m)\). Fix a cube \(Q = Q(x_0, d)\) and \(\bar{x} \in Q\). Write, by \(T_1(g) = 0\),
\[ S_b(f)(x) = \sum_{k=1}^{m} T^{k,3}M_{b-b_{2Q}}T^{k,4}(f)(x) + \sum_{k=1}^{m} T^{k,5}I_{\alpha}M_b T^{k,6}(f)(x) \]
\[ = A_b(x) + B_b(x) = A_{b-b_{2Q}}(x) + B_{b-b_{2Q}}(x), \]
where
\[ A_{b-bQ}(x) = \sum_{k=1}^{m} T^{k,3} M_{(b-bQ)\chi_{2Q}} I_{\alpha} T^{k,4}(f)(x) + \sum_{k=1}^{m} T^{k,3} M_{(b-bQ)\chi_{2Q}^c} I_{\alpha} T^{k,4}(f)(x) \]
\[ = A_1(x) + A_2(x) \]
and
\[ B_{b-bQ}(x) = \sum_{k=1}^{m} T^{k,5} I_{\alpha} M_{(b-bQ)\chi_{2Q}} T^{k,6}(f)(x) + \sum_{k=1}^{m} T^{k,5} I_{\alpha} M_{(b-bQ)\chi_{2Q}^c} T^{k,6}(f)(x) \]
\[ = B_1(x) + B_2(x). \]

Then
\[ \left( \frac{1}{|Q|} \int_{Q} |A_{b-bQ}(f)(x) - A_2(x_0)|^\eta \, dx \right)^{1/\eta} \leq C \left( \frac{1}{|Q|} \int_{Q} |A_1(x)|^\eta \, dx \right)^{1/\eta} + C \left( \frac{1}{|Q|} \int_{Q} |A_2(x) - A_2(x_0)|^\eta \, dx \right)^{1/\eta} \]
\[ = I_1 + I_2 \]

and
\[ \left( \frac{1}{|Q|} \int_{Q} |B_{b-bQ}(f)(x) - B_2(x_0)|^\eta \, dx \right)^{1/\eta} \leq C \left( \frac{1}{|Q|} \int_{Q} |B_1(x)|^\eta \, dx \right)^{1/\eta} + C \left( \frac{1}{|Q|} \int_{Q} |B_2(x) - B_2(x_0)|^\eta \, dx \right)^{1/\eta} \]
\[ = I_3 + I_4. \]

For \( I_1 \) and \( I_2 \), by using the same argument as in the proof of Theorem 1, we get
\[ I_1 \leq C \sum_{k=1}^{m} \left( \frac{1}{|Q|} \int_{R^n} |T^{k,3} M_{(b-bQ)\chi_{2Q}} I_{\alpha} T^{k,4}(f)(x)|^\eta \, dx \right)^{1/\eta} \]
\[ \leq \sum_{k=1}^{m} |Q|^{-1} \left\| T^{k,3} M_{(b-bQ)\chi_{2Q}} I_{\alpha} T^{k,4}(f) \chi_{\Omega} \right\|_{L^\eta} \]
\[ \leq C \sum_{k=1}^{m} |Q|^{-1} \left\| M_{(b-bQ)\chi_{2Q}} I_{\alpha} T^{k,4}(f) \right\|_{W^1 L^1} \]
\[ \leq C \sum_{k=1}^{m} |Q|^{-1} \left\| M_{(b-bQ)\chi_{2Q}} I_{\alpha} T^{k,4}(f) \right\|_{L^1} \]
\[ \leq C \sum_{k=1}^{m} \left\| b(x) - b_{2Q} \right\|_{L^1} \left\| I_{\alpha} T^{k,4}(f) \right\|_{L^1} \]
\[ \leq C \sum_{k=1}^{m} \left\| b - b_{2Q} \right\|_{L^1} \left\| I_{\alpha} T^{k,4}(f) \right\|_{L^1} \]
\[ \leq C \left\| b \right\|_{BMO} \sum_{k=1}^{m} \left\| I_{\alpha} T^{k,4}(f) \right\|_{L^1} \]
\[ \leq C \left\| b \right\|_{BMO} \sum_{k=1}^{m} \left( I_{\alpha} T^{k,4}(f)(\bar{x}) \right), \]
\[ I_2 \leq C \sum_{k=1}^{m} \frac{1}{|Q|} \int_{Q} \sum_{j=1}^{\infty} \int_{2^j d \leq |x-y| < 2^{j+1} d} |b(y) - b_{2Q}| |K(x, x-y) - K(x_0, x_0-y)| \times |\alpha T^{k,4}_\alpha(f)(y)| dy dx \]
\[ \leq C \sum_{k=1}^{m} \frac{1}{|Q|} \int_{Q} \sum_{j=1}^{\infty} \int_{2^j d \leq |x-y| < 2^{j+1} d} |b(y) - b_{2Q}| \left| \frac{x-x_0}{|x_0-y|^{n+1}} \right| |\alpha T^{k,4}_\alpha(f)(y)| dy dx \]
\[ \leq C \sum_{k=1}^{m} \sum_{j=1}^{\infty} 2^{-j} \frac{1}{|2^{j+1} Q|} \int_{2^{j+1} Q} |b(y) - b_{2Q}| |\alpha T^{k,4}_\alpha(f)(y)| dy \]
\[ \leq C \sum_{k=1}^{m} \sum_{j=1}^{\infty} 2^{-j} \left| b - b_{2Q} \right|_{exp, 2^{j+1} Q} ||\alpha T^{k,4}_\alpha(f)||_{L(log L), 2^{j+1} Q} \]
\[ \leq C \sum_{k=1}^{m} \sum_{j=1}^{\infty} j2^{-j} ||b||_{BMO} M^2(\alpha T^{k,4}_\alpha(f))(\bar{x}) \]
\[ \leq C ||b||_{BMO} \sum_{k=1}^{m} M^2(\alpha T^{k,4}_\alpha(f))(\bar{x}). \]

For \( I_3 \), by Lemma 4, we get, for \( 1 < p < \min(s, n/\alpha) \) and \( 1/q = 1/p - \alpha/n \),
\[ I_3 \leq C \sum_{k=1}^{m} \left( \frac{1}{|Q|} \int_{R^n} |\alpha M_{b-b_Q} x_{2Q} T^{k,6}_\alpha(f)(x)|^q dx \right)^{1/q} \]
\[ \leq C \sum_{k=1}^{m} |Q|^{-1/q} \left[ \int_{2Q} (|b(x) - b_Q||T^{k,6}_\alpha(f)(x)|)^p dx \right]^{1/p} \]
\[ \leq C \sum_{k=1}^{m} \left( \frac{1}{|2Q|} \int_{2Q} |b(x) - b_Q|^{ps/(s-p)} dx \right)^{(s-p)/ps} \left( \frac{1}{|2Q|^{1-s\alpha/n}} \int_{2Q} |T^{k,6}_\alpha(f)(x)|^s dx \right)^{1/s} \]
\[ \leq C ||b||_{BMO} \sum_{k=1}^{m} M_{\alpha, s}(T^{k,6}_\alpha(f))(\bar{x}). \]

For \( I_4 \), we get
\[ I_4 \leq |Q|^{-1} \sum_{k=1}^{m} \left[ \int_{Q} \left( \int_{2Q} |b(y) - b_{2Q}| \left| \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x_0-y|^{n-\alpha}} \right| |T^{k,6}_\alpha(f)(y)| dy \right) dx \right] \]
\[ \leq C \sum_{k=1}^{m} \sum_{j=1}^{\infty} \int_{2^j d \leq |x-y_0| < 2^{j+1} d} |b(y) - b_{2Q}| \frac{d}{|x_0-y|^{n-\alpha+1}} |T^{k,6}_\alpha(f)(y)| dy \]
\[ \leq C \sum_{k=1}^{m} \sum_{j=1}^{\infty} d(2^j d)^{-n+\alpha-1}(2^{j+1} d)^{(n-1)/s}(2^j d)^{n/s-\alpha} \]
\[ \times \left( \frac{1}{|2^{j+1} Q|} \int_{2^{j+1} Q} |b(y) - b_{2Q}|^{s'} dy \right)^{1/s'} \left( \frac{1}{|2^{j+1} Q|^{1-s\alpha/n}} \int_{2^{j+1} Q} |T^{k,6}_\alpha(f)(y)|^s dy \right)^{1/s} \]
\[ \leq C \|b\|_{BMO} \sum_{k=1}^{m} M_{\alpha,s}(T^{k,6}(f))(\tilde{x}) \sum_{j=1}^{\infty} j 2^{-j} \]
\[ \leq C \|b\|_{BMO} \sum_{k=1}^{m} M_{\alpha,s}(T^{k,6}(f))(\tilde{x}) . \]

This completes the proof of Theorem 2. \qed

**Proof of Theorem 3.** By Theorem 1 and Lemmas 7–8, we have
\[ \|T_b(f)\|_{L^p,\varphi} \leq \|M_\eta(T_b(f))\|_{L^p,\varphi} \leq C \|\{T_b(f)\}_\eta\|_{L^p,\varphi} \]
\[ \leq C \|b\|_{BMO} \sum_{k=1}^{m} \|M^2(T^{k,2}(f))\|_{L^p,\varphi} \]
\[ \leq C \|b\|_{BMO} \sum_{k=1}^{m} \|T^{k,2}(f)\|_{L^p,\varphi} \]
\[ \leq C \|b\|_{BMO} \|f\|_{L^p,\varphi} . \]

This completes the proof of Theorem 3. \qed

**Proof of Theorem 4.** Choose \(1 < s < p\) in Theorem 2, then we have, by Lemmas 7 and 8,
\[ \|S_b(f)\|_{L^q,\varphi} \leq \|M_\eta(S_b(f))\|_{L^q,\varphi} \leq C \|\{S_b(f)\}_\eta\|_{L^q,\varphi} \]
\[ \leq C \|b\|_{BMO} \sum_{k=1}^{m} (\|M^2(I_{\alpha}T^{k,4}(f))\|_{L^q,\varphi} + \|M_{\alpha,s}(T^{k,6}(f))\|_{L^q,\varphi}) \]
\[ \leq C \|b\|_{BMO} \sum_{k=1}^{m} (\|I_{\alpha}T^{k,4}(f)\|_{L^q,\varphi} + \|T^{k,6}(f)\|_{L^p,\varphi}) \]
\[ \leq C \|b\|_{BMO} \sum_{k=1}^{m} (\|T^{k,4}(f)\|_{L^p,\varphi} + \|f\|_{L^p,\varphi}) \]
\[ \leq C \|b\|_{BMO} \|f\|_{L^p,\varphi} . \]

This completes the proof of Theorem 4. \qed

**REFERENCES**


