$M^K$-TYPE ESTIMATES FOR MULTILINEAR COMMUTATOR OF SINGULAR INTEGRAL OPERATOR WITH GENERAL KERNEL

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ABSTRACT. In this paper, we prove the $M^K$-type inequality for multilinear commutator related to generalized singular integral operator. By using the $M^K$-type inequality, we obtain the weighted $L^p$-norm inequality and the weighted estimate on the generalized Morrey spaces for the multilinear commutator.

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1. Introduction and Preliminaries

Let $b \in BMO(\mathbb{R}^n)$ and $T$ be the Calderón-Zygmund operator. Consider the commutator defined by

$$[b, T](f) = bT(f) - T(bf).$$

As the development of singular integral operators (see [5][16]), their commutators have been well studied. In [4][13][14][15], the authors prove that the commutators generated by the singular integral operators and $BMO$ functions are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Chanillo (see [2]) proves a similar result when singular integral operators are replaced by the fractional integral operators. In this paper, we will study some singular integral operators as following (see [1][8]).

Definition 1. Let $T : S \to S'$ be a linear operator such that $T$ is bounded on $L^2(\mathbb{R}^n)$ and there exists a locally integrable function $K(x, y)$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$ such that

$$T(f)(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy$$

for every bounded and compactly supported function $f$, where $K$ satisfies: there is a sequence of positive constant numbers $\{C_k\}$ such that for any $k \geq 1$,

$$\int_{2|y-z| < |x-y|} (|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)|) dx \leq C,$$
and
\[
\left( \int_{2^k|z-y| \leq |x-y| < 2^{k+1}|z-y|} \left( |K(x, y) - K(x, z)| + |K(y, x) - K(z, x)| \right)^q dy \right)^{1/q} \leq C_k (2^k|z-y|)^{-n/q'},
\]
where \(1 < q' < 2\) and \(1/q + 1/q' = 1\).

Suppose \(b_j \ (j = 1, \cdots, m)\) are the fixed locally integrable functions on \(\mathbb{R}^n\). The multilinear commutator of the singular integral operator is defined by
\[
T_{\vec{b}}(f)(x) = \int_{\mathbb{R}^n} \prod_{j=1}^m (b_j(x) - b_j(y)) K(x, y) f(y) dy.
\]

Note that the classical Calderón-Zygmund singular integral operator satisfies Definition 1 with \(C_j = 2^{-j\delta}\) (see [5][16]).

Also note that when \(m = 1\), \(T_{\vec{b}}\) is just the commutator what we mentioned above. It is well known that multilinear operator are of great interest in harmonic analysis and have been widely studied by many authors (see [13-14]). In [15], Pérez and Trujillo-Gonzalez prove a sharp estimate for the multilinear commutator. The purpose of this paper has two-fold, first, we establish a \(M^k\)-type estimate for the multilinear commutator related to the generalized singular integral operators, and second, we obtain the weighted \(L^p\)-norm inequality and the weighted estimates on the generalized Morrey space for the multilinear commutator by using the \(M^k\)-type inequality.

**Definition 2.** Let \(\varphi\) be a positive, increasing function on \(\mathbb{R}^+\) and there exists a constant \(D > 0\) such that
\[
\varphi(2t) \leq D\varphi(t) \quad \text{for} \quad t \geq 0.
\]
Let \(w\) be a non-negative weight function on \(\mathbb{R}^n\) and \(f\) be a locally integrable function on \(\mathbb{R}^n\). Set, for \(1 \leq p < \infty\),
\[
||f||_{L^p,\varphi(w)} = \sup_{x \in \mathbb{R}^n, \ d > 0} \left( \frac{1}{\varphi(d)} \int_{Q(x, d)} |f(y)|^p w(y) dy \right)^{1/p},
\]
where \(Q(x, d) = \{ y \in \mathbb{R}^n : |x - y| < d \}\). The generalized weighted Morrey space is defined by
\[
L^{p,\varphi}(\mathbb{R}^n, w) = \{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : ||f||_{L^p,\varphi(w)} < \infty \}.
\]
If \(\varphi(d) = d^\delta, \ \delta > 0\), then \(L^{p,\varphi}(\mathbb{R}^n, w) = L^{p,\delta}(\mathbb{R}^n, w)\), which is the classical Morrey spaces (see [11][12]). If \(\varphi(d) = 1\), then \(L^{p,\varphi}(\mathbb{R}^n, w) = L^p(w)\), which is the weighted Lebesgue spaces (see [5]).
As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [3][6][7][9][10]).

Now, let us introduce some notations. Throughout this paper, $Q$ will denote a cube of $\mathbb{R}^n$ with sides parallel to the axes. For any locally integrable function $f$, the sharp maximal function of $f$ is defined by

$$(f^\#(x)) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x) dx$. It is well-known that (see [5][16])

$$(f^\#(x)) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

We say that $f$ belongs to $BMO(\mathbb{R}^n)$ if $f^\#$ belongs to $L^\infty(\mathbb{R}^n)$ and define $||f||_{BMO} = ||f^\#||_{L^\infty}$.

Let

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

For $0 < p < \infty$, we denote $M_p f(x)$ by

$$M_p(f)(x) = \left[ M(|f|^p)(x) \right]^{1/p}.$$

For $k \in \mathbb{N}$, we denote by $M^k$ the operator $M$ iterated $k$ times, i.e. $M^1(f)(x) = M(f)(x)$ and $M^k(f)(x) = M(M^{k-1}(f))(x)$ when $k \geq 2$.

Let $\Phi$ be a Young function and $\tilde{\Phi}$ be the complementary associated to $\Phi$, we denote that the $\Phi$-average by, for a function $f$,

$$||f||_{\Phi,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi \left( \frac{|f(y)|}{\lambda} \right) d(y) \leq 1 \right\}$$

and the maximal function associated to $\Phi$ by

$$M_{\Phi}(f)(x) = \sup_{z \in Q} ||f||_{\Phi,Q}.$$

The Young functions to be using in this paper are $\Phi(t) = t(1 + \log t)^r$ and $\tilde{\Phi}(t) = \exp(t^{1/r})$, the corresponding average and maximal functions denoted by $||\cdot||_{L(\log L)^r,B}$,

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and any\( j = 1 \), we know the generalized Hölder’s inequality:
\[
\frac{1}{|Q|} \int_Q |f(y)g(y)| \, dy \leq \|f\|_{\dot{\Phi}(Q)} \|g\|_{\dot{\Phi}(Q)}.
\]

And we can also obtain the following inequalities:
\[
\|f\|_{L((logL)^1/r)} \leq M_{L((logL)^{1/r})}(f) \leq CM_{L((logL)^m)}(f) \leq CM^{m+1}(f),
\]
\[
\|b - b_Q\|_{expL^r} \leq C\|b\|_{BMO},
\]
\[
|b_{2^{k+1}Q} - b_{2Q}| \leq Ck\|b\|_{BMO}.
\]

for \( r, r_j \geq 1, j = 1, 2, \ldots, m \) with \( 1/r = 1/r_1 + 1/r_2 \cdots + 1/r_m \), and \( b \in BMO(R^n) \).

Given a positive integer \( m \) and \( 1 \leq j \leq m \), we denote by \( C_j^m \) the family of all finite subsets \( \sigma = \{\sigma(1), \cdots, \sigma(j)\} \) of \( \{1, \cdots, m\} \) of \( j \) different elements and \( \sigma(i) < \sigma(j) \) when \( i < j \). For \( \sigma \in C_j^m \), set \( \sigma^0 = \{1, \cdots, m\} \setminus \sigma \). For \( \vec{b} = (b_1, \cdots, b_m) \) and \( \sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m \), set \( \vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)}) \), \( b_\sigma = \prod_{i=1}^j b_{\sigma(i)} \) and
\[
\|\vec{b}_\sigma\|_{BMO} = \prod_{i=1}^j \|b_{\sigma(i)}\|_{BMO}.
\]

We denote the Muckenhoupt weights by \( A_p \) for \( 1 \leq p < \infty \)(see [5]), that is
\[
A_1 = \{ w : M(w)(x) \leq Cw(x), a.e. \}
\]

and
\[
A_p = \left\{ w : \sup_Q \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} \, dx \right)^{p-1} < \infty \right\}, \quad 1 < p < \infty.
\]

2. THEOREMS AND PROOFS

Now we give some theorems as following.

**Theorem 1.** Let \( T \) be the singular integral operator as Definition 1, the sequence \( \{kmC_k\} \in l^1, \ q' \leq s < \infty, \ 0 < r < 1, \ k \geq m + 1, \ k \in N \) and \( b_j \in BMO(R^n) \) for \( j = 1, \cdots, m \). Then there exists a constant \( C > 0 \) such that for any \( f \in C_0^\infty(R^n) \) and any \( \vec{x} \in R^n \),
\[
(T_{\vec{b}}(f))^s_{pr}(\vec{x}) \leq C\|\vec{b}\|_{BMO} \left( M^k(f)(\vec{x}) + \sum_{j=1}^m \sum_{\sigma \in C_j^m} M^k(T_{\vec{b}_\sigma}(f))(\vec{x}) + M_s(f)(\vec{x}) \right).
\]
**Theorem 2.** Let $T$ be the singular integral operator as Definition 1, the sequence \( \{k^m C_k\} \in l^1 \), \( q' \leq p < \infty \), \( w \in A_p \) and \( b_j \in BMO(R^n) \) for \( j = 1, \ldots, m \). Then \( T_{\vec{b}} \) is bounded on \( L^p(w) \).

**Theorem 3.** Let $T$ be the singular integral operator as Definition 1, the sequence \( \{k^m C_k\} \in l^1 \), \( q' \leq p < \infty \), \( w \in A_1 \) and \( b_j \in BMO(R^n) \) for \( j = 1, \ldots, m \). Then, if \( 0 < D < 2^n \),

\[
\|T_{\vec{b}} f\|_{L^p(w)} \leq C\|\vec{b}\|_{BMO} \|f\|_{L^p(w)}.
\]

In order to better proof of the theorem above, we need the following lemmas

**Lemma 1.** Let \( 1 < r < \infty \) and \( b_j \in BMO(R^n) \) with \( j = 1, \ldots, k \) and \( k \in \mathbb{N} \). Then, we have

\[
\frac{1}{|Q|} \int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q| dy \leq C \prod_{j=1}^k |b_j|_{BMO},
\]

\[
\left( \frac{1}{|Q|} \int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q|^{r/d} dy \right)^{1/r} \leq C \prod_{j=1}^k |b_j|_{BMO}.
\]

Similarly, for \( \sigma \in C_{m_k} \) when \( k \leq m \) and \( m \in \mathbb{N} \), we have:

\[
\frac{1}{|Q|} \int_Q |(b(y) - (b_j)_{Q_\sigma}| dy \leq C|b_\sigma|_{BMO}
\]

and

\[
\left( \frac{1}{|Q|} \int_Q |(b(y) - (b_j)_{Q_\sigma}|^{r/d} dy \right)^{1/r} \leq C|b_\sigma|_{BMO}.
\]

In fact, we just need to choose \( p_j > 1 \) and \( q_j > 1 \), where \( 1 \leq j \leq k \), such that \( 1/p_1 + \cdots + 1/p_k = 1 \) and \( r/q_1 + \cdots + r/q_k = 1 \). After that, using the Hölder’s inequality with exponent \( 1/p_1 + \cdots + 1/p_k = 1 \) and \( r/q_1 + \cdots + r/q_k = 1 \), respectively, we may get the results.

**Lemma 2.** ([5, p.485]) Let \( 0 < p < q < \infty \) and for any function \( f \geq 0 \). We define that, for \( 1/r = 1/p - 1/q \)

\[
\|f\|_{WL^q} = \sup_{\lambda > 0} \lambda |\{x \in R^n : f(x) > \lambda\}|^{1/q}, \quad N_{p,q}(f) = \sup_E \|f\chi_E\|_{L^p}/\|\chi_E\|_{L^r},
\]

where the sup is taken for all measurable sets \( E \) with \( 0 < |E| < \infty \). Then

\[
\|f\|_{WL^q} \leq N_{p,q}(f) \leq (q/(q-p))^{1/p} \|f\|_{WL^q}.
\]
Lemma 3. (see [5]) Let $0 < p, \eta < \infty$ and $w \in \bigcup_{1 \leq r < \infty} A_r$. Then
\[
\|M_\eta(f)\|_{L^p(w)} \leq C\|f^\#(f)\|_{L^p(w)}.
\]

Lemma 4. Let $1 < p < \infty$, $1 \leq q < p$ and $w \in A_1$. Then, if $0 < D < 2^n$,
\[
\|M_q(f)\|_{L^{p,q}(w)} \leq C\|f\|_{L^{p,q}(w)}.
\]

Proof. Let $f \in L^{p,q}(R^n, w)$. Note that $1 \leq q < p$ and for any $w \in A_1$,
\[
\int_{R^n} |M_q(f)(y)|^p w(y)dy \leq C \int_{R^n} |f(y)|^p w(y)dy.
\]
For a cube $Q = Q(x, d) \subset R^n$, we get
\[
\int_Q |M_q(f)(y)|^p w(y)dy \\
\leq \int_{R^n} |M_q(f)(y)|^p M(w\chi_Q)(y)dy \\
\leq C \int_{R^n} |f(y)|^p M(w\chi_Q)(y)dy \\
= C \left[ \int_Q |f(y)|^p M(w\chi_Q)(y)dy + \sum_{k=0}^{\infty} \int_{2^{k+1}Q\setminus 2^kQ} |f(y)|^p M(w\chi_Q)(y)dy \right] \\
\leq C \left[ \int_Q |f(y)|^p w(y)dy + \sum_{k=0}^{\infty} \frac{\int_{2^{k+1}Q\setminus 2^kQ} |f(y)|^p w(y)dy}{2^{2k+1}|Q|} \right] \\
\leq C \left[ \int_Q |f(y)|^p w(y)dy + \sum_{k=0}^{\infty} \frac{\int_{2^{k+1}Q} |f(y)|^p w(y)dy}{2^{2k}|Q|^{\frac{n}{n-1}}} \right] \\
\leq C \left[ \int_Q |f(y)|^p w(y)dy + \sum_{k=0}^{\infty} \int_{2^{k+1}Q} |f(y)|^p \frac{w(y)}{2^{2k}|Q|^{\frac{n}{n-1}}} dy \right] \\
\leq C\|f\|_{L^{p,q}(w)}^{\frac{p}{p-q}} \sum_{k=0}^{\infty} 2^{-nk}\varphi(2^{k+1}d) \\
\leq C\|f\|_{L^{p,q}(w)}^{\frac{p}{p-q}} \sum_{k=0}^{\infty} (2^{-n}D)^k \varphi(d) \\
\leq C\|f\|_{L^{p,q}(w)}^{\varphi(d)},
\]
thus
\[
\|M_q(f)\|_{L^{p,q}(w)} \leq C\|f\|_{L^{p,q}(w)}.
\]
Lemma 5. Let $1 < p < \infty$, $0 < D < 2^n$, $w \in A_1$. Then, for $f \in L^{p,\varphi}(R^n, w)$,
\[ \|M(f)\|_{L^{p,\varphi}(w)} \leq C\|f\|_{L^{p,\varphi}(w)}. \]

Lemma 6. Let $T$ be the bounded linear operators on $L^q(R^n, w)$ for any $1 < q < \infty$ and $w \in A_1$. Then, for $1 < p < \infty$, $w \in A_1$ and $0 < D < 2^n$,
\[ \|T(f)\|_{L^{p,\varphi}(w)} \leq C\|f\|_{L^{p,\varphi}(w)}. \]

The proofs of two Lemmas are similar to that of Lemma 4, we omit the details.

Proof of Theorem 1. It suffices to prove for $f \in C^\infty_0(R^n)$ and some constant $C_0$, the following inequality holds:
\[ \left( \frac{1}{|Q|} \int_Q \left| T^*_Q(f)(x) - C_0 \right|^r dx \right)^{1/r} \leq C\|\tilde{b}\|_{BMO} \left( M^k(f)(\tilde{x}) + \sum_{j=1}^m \sum_{\sigma \in C^m_j} M^k(T^*_{\tilde{b},\sigma}(f))(\tilde{x}) \right). \]

Fix a ball $Q = Q(x_0, d)$ and $\tilde{x} \in Q$, we write $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$. Following [20], we will consider the cases $m = 1$ and $m > 1$, and choose $C_0 = T(((b_1)_{2Q} - b_1)f_2)(x_0)$ and $C_0 = T(\prod_{j=1}^m (b_j - (b_j)_{2Q})f_2)(x_0)$, respectively.

We first consider the Case $m = 1$. For $C_0 = T(((b_1)_{2Q} - b_1)f_2)(x_0)$, we write
\[ T_{b_1}(f)(x) = (b_1(x) - (b_1)_{2Q})T(f)(x) - T((b_1 - (b_1)_{2Q})f)(x). \]

Then
\[ |T_{b_1}(f)(x) - C_0| = \left| (b_1(x) - (b_1)_{2Q})T(f)(x) + T(((b_1)_{2Q} - b_1)f)(x) - T((b_1)_{2Q} - b_1)f_2)(x_0) \right| \]
\[ \leq \left| (b_1(x) - (b_1)_{2Q})T(f)(x) \right| + \left| T(((b_1)_{2Q} - b_1)f_1)(x) \right| \]
\[ + \left| T((b_1)_{2Q} - b_1)f_2)(x) - T((b_1)_{2Q} - b_1)f_2)(x_0) \right| \]
\[ = A(x) + B(x) + C(x). \]

For $A(x)$, we get
\[ \left( \frac{1}{|Q|} \int_Q |A(x)|^r dx \right)^{1/r} \leq \frac{1}{|Q|} \int_Q |A(x)| dx \]
\[ \leq \frac{1}{|Q|} \int_Q \left| (b_1(x) - (b_1)_{2Q})T(f)(x) \right| dx \]
\[ \leq \|b_1 - (b_1)_{2Q}\|_{L,2Q} \|T(f)\|_{L(\log L),2Q} \]
\[ \leq C\|b_1\|_{BMO} M^2(T(f))(\tilde{x}). \]
For $B(x)$, by the weak type $(1,1)$ of $T$ and Lemma 2, we obtain

$$\left(\frac{1}{|Q|} \int_Q |B(x)|^r dx\right)^{1/r} \leq \frac{1}{|Q|} \int_Q |B(x)| dx \leq \frac{1}{|Q|} \int_Q |T((b_1)_{2Q} - b_1)f_1(x)| dx \leq \left(\frac{1}{|Q|} \int_{2Q} |T((b_1 - (b_1)_{2Q})f\chi_{2Q}(x))|^p dx\right)^{1/p} \leq \frac{1}{|Q|} \frac{1}{|Q|^{1/p-1}} \|T((b_1 - (b_1)_{2Q})f\chi_{2Q})\|_{L^p} \leq \frac{C}{|Q|} \|T((b_1 - (b_1)_{2Q})f\chi_{2Q})\|_{W^{1,1}} \leq \frac{C}{|Q|} \|((b_1 - (b_1)_{2Q})f\chi_{2Q})\|_{L^1} \leq \frac{C}{|Q|} \int_{2Q} |b_1(x) - (b_1)_{2Q}| |f(x)| dx \leq C|b_1 - (b_1)_{2Q}|_{\exp L, 2Q} \|f\|_{L(\log L), 2Q} \leq C|b_1|_{BMO} M^2(f) (\tilde{x})$.

For $C(x)$, recalling that $s > q'$, taking $1 < p < \infty$, $1 < t < s$ with $1/p + 1/q + 1/t = 1$, by the Hölder’s inequality, we have, for $x \in Q$,

$$|T((b_1 - (b_1)_{2Q})f_2)(x) - T((b_1 - (b_1)_{2Q})f_2)(x_0)| = \left|\int_{(2Q)^c} (b_1(y) - (b_1)_{2Q})f(y)(K(x, y) - K(x_0, y)) dy\right| \leq \sum_{k=1}^{\infty} \int_{2^k|x-x_0| \leq |y-x_0| < 2^{k+1}|x-x_0|} |K(x, y) - K(x_0, y)||f(y)||b_1(y) - (b_1)_{2Q}| dy \leq C \sum_{k=1}^{\infty} \left(\int_{2^k|x-x_0| \leq |y-x_0| < 2^{k+1}|x-x_0|} |K(x, y) - K(x_0, y)| dy\right)^{1/q} \times \left(\int_{|y-x_0| < 2^{k+1}|x-x_0|} |b_1(y) - (b_1)_{2Q}|^p dy\right)^{1/p} \left(\int_{|y-x_0| < 2^{k+1}|x-x_0|} |f(y)|^t dy\right)^{1/t} \leq C \sum_{k=1}^{\infty} C_k \left(\frac{2^{k+1}Q}{(2^k d)^{n/q'}}\right)^{1/p+1/t} |b_1|_{BMO} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(y)|^n dy\right)^{1/s}.$$
\[ \leq C ||b_1||_{BMO} \sum_{k=1}^{\infty} kC_k M_s(f)(\bar{x}) \]
\[ \leq C ||b_1||_{BMO} M_s(f)(\bar{x}), \]
thus
\[ \left( \frac{1}{|Q|} \right) \int_Q |C(x)|^r dx \right)^{1/r} \leq C ||b_1||_{BMO} M_s(f)(\bar{x}). \]

Now, we consider the Case \( m \geq 2 \). we have, for \( b = (b_1, \cdots, b_m) \),
\[
T_b(f)(x) = \int_{R^n} \prod_{j=1}^{m} (b_j(x) - b_j(y))K(x, y)f(y)dy
\]
\[ = \int_{R^n} \prod_{j=1}^{m} [(b_j(x) - (b_j)_{2Q}) - (b_j(y) - (b_j)_{2Q})]K(x, y)f(y)dy
\]
\[ = \sum_{j=0}^{m} \sum_{\sigma \in C_j} (-1)^{m-j}(b(x) - (b)_{2Q})_\sigma \int_{R^n} (b(y) - (b)_{2Q})_\sigma K(x, y)f(y)dy
\]
\[ = \prod_{j=1}^{m} (b_j(x) - (b_j)_{2Q})T(f)(x) + (-1)^{m}T(\prod_{j=1}^{m} (b_j - (b_j)_{2Q})f)(x)
\]
\[ + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j} (-1)^{m-j}((b_j(x) - (b_j)_{2B})_\sigma T(b_j - (b_j)_{2B})_\sigma f)(x)
\]
thus, recall that \( C_0 = T(\prod_{j=1}^{m} (b_j - (b_j)_{2B})f_2)(x_0), \)
\[ |T_b(f)(x) - T(\prod_{j=1}^{m} (b_j - (b_j)_{2B})f_2)(x_0)|
\]
\[ \leq |\prod_{j=1}^{m} (b_j(x) - (b_j)_{2Q})T(f)(x)|
\]
\[ + |T(\prod_{j=1}^{m} (b_j - (b_j)_{2Q})f_1)(x)|
\]
\[ + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j} ((b_j(x) - (b_j)_{2Q})_\sigma T(b_j - (b_j)_{2Q})_\sigma f)(x)|
\]
\[ + |T(\prod_{j=1}^{m} (b_j - (b_j)_{2Q}) f_2)(x)| - T(\prod_{j=1}^{m} (b_j - (b_j)_{2Q}) f_2)(x_0)| = I_1(x) + I_2(x) + I_3(x) + I_4(x). \]

For \( I_1(x) \), we get,
\[
\left( \frac{1}{|Q|} \int_{Q} |I_1(x)|^r \, dx \right)^{1/r} \leq \frac{1}{|Q|} \int_{Q} |I_1(x)| \, dx
\]
\[
\leq \frac{1}{|Q|} \int_{Q} \prod_{j=1}^{m} |(b_j(x) - (b_j)_{2Q})||T(f)(x)| \, dx
\]
\[
\leq C \prod_{j=1}^{m} \| (b_j - (b_j)_{2Q}) \|_{L^{r/2}} \| T(f) \|_{L(\log L)^{r/2}}
\]
\[
\leq C \prod_{j=1}^{m} \| b_j \|_{BMO} M^{m+1}(T(f))(\tilde{x})
\]
\[
\leq C \| \tilde{b} \|_{BMO} M^k(T(f))(\tilde{x}).
\]

For \( I_2(x) \), by the boundness of \( T \) on \( L^p(\mathbb{R}^n) \) and similar to the proof of \( B(x) \), using Lemma 2, we get
\[
\left( \frac{1}{|Q|} \int_{Q} |I_2(x)|^r \, dx \right)^{1/r} \leq \frac{1}{|Q|} \int_{Q} |I_2(x)| \, dx
\]
\[
= \left( \frac{1}{|Q|} \int_{Q} |T(\prod_{j=1}^{m} (b_j(y) - (b_j)_{2Q}) f_1)(x)| \, dx \right)^{1/p}
\]
\[
\leq \left( \frac{1}{|Q|} \int_{Q} |T(\prod_{j=1}^{m} (b_j - (b_j)_{2Q}) f_1)(x)|^p \, dx \right)^{1/p}
\]
\[
= \frac{1}{|Q|} \frac{1}{|Q|} \prod_{j=1}^{m} \| (b_j - (b_j)_{2Q}) f_1 \|_{L^p}
\]
\[
\leq \frac{1}{|Q|} \| T(\prod_{j=1}^{m} (b_j - (b_j)_{2Q}) f_1) \|_{W_{L^p}}
\]
\[
\leq \frac{1}{|Q|} \| \prod_{j=1}^{m} (b_j - (b_j)_{2Q}) f_1 \|_{L^1}
\]
\[
\leq \frac{1}{|Q|} \int_B \prod_{j=1}^{m} |(b_j(x) - (b_j)_{2Q})| f_1(x) \, dx
\]
\begin{align*}
\leq \quad & C \prod_{j=1}^{m} \|(b_j - (b_j)_{2Q})\|_{\exp L^{1/r_j}, 2Q} \|f\|_{L(\log L)^{r}, 2Q} \\
\leq \quad & C \|\tilde{b}\|_{BMO} M^{m+1}(f)(\tilde{x}) \\
\leq \quad & C \|\tilde{b}\|_{BMO} M^{k}(f)(\tilde{x}).
\end{align*}

For $I_3(x)$, by Lemma 2,

\[
\left( \frac{1}{|Q|} \int_{Q} |I_3(x)|^r \, dx \right)^{1/r} \leq \frac{1}{|Q|} \int_{Q} |I_3(x)| \, dx
\]

\[
\leq \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|} \int_{Q} |(b_j(x) - (b_j)_{2Q})_\sigma |T(b_j - (b_j)_{2Q})_\sigma (f)(x)| \, dx
\]

\[
\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} |(b_j(x) - (b_j)_{2Q})_\sigma |_{\exp L^{1/r_j}, 2Q} \|T(b_j - (b_j)_{2Q})_\sigma (f)\|_{L(\log L)^{r}, 2Q}
\]

\[
\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} |b_\sigma|_{BMO} M^{m+1}(T_{b_\sigma}(f))(\tilde{x})
\]

\[
\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \|\tilde{b}\|_{BMO} M^{k}(T_{b_\sigma}(f))(\tilde{x}).
\]

For $I_4(x)$, similar to the proof of $C(x)$ in the Case $m = 1$, for $1 < p < \infty, 1 < t < s$ with $1/p + 1/q + 1/t = 1$, we have

\[
|T((\prod_{j=1}^{m} (b_j - (b_j)_{2Q}) f_2)(x) - T((\prod_{j=1}^{m} (b_j - (b_j)_{2Q}) f_2)(x_0))|
\]

\[
\leq C \sum_{k=1}^{\infty} \left( \int_{2^k|x-x_0| \leq |y-x_0| < 2^{k+1}|x-x_0|} |(K(x,y) - K(x_0,y))| |f(y)| \prod_{j=1}^{m} |(b_j(y) - (b_j)_{2Q})| \, dy \right)^{1/q}
\]

\[
\times \left( \int_{|y-x_0| < 2^{k+1}|x-x_0|} \prod_{j=1}^{m} |b_j(y) - (b_j)_{2Q}|^p \, dy \right)^{1/p} \left( \int_{|y-x_0| < 2^{k+1}|x-x_0|} |f(y)|^t \, dy \right)^{1/t}
\]

\[
\leq C \sum_{k=1}^{\infty} C_k \left( \frac{2^{k+q} |Q|^{1/p+1/t}}{(2^k d)^{q/n'}} \right)^{k} \prod_{j=1}^{m} \|b_j\|_{BMO} \left( \frac{1}{|2^k+1|Q} \int_{2^k+1|Q} |f(y)|^s \, dy \right)^{1/s}
\]

\[
\leq C \|\tilde{b}\|_{BMO} \sum_{k=1}^{\infty} k^m C_k M_s(f)(\tilde{x})
\]
\[ \leq C||\vec{b}||_{BMO}M_s(f)(\tilde{x}), \]

thus
\[ \left(\frac{1}{|Q|}\int_Q |I_4(x)|^r dx\right)^{1/r} \leq ||\vec{b}||_{BMO}M_s(f)(\tilde{x}). \]

This completes the proof of the theorem.

**Proof of Theorem 2.** Choose \( q' < s < p \) in Theorem 1, by the \( L^p(w) \)-boundedness of \( M^k \) and \( M_s \), we may obtain the conclusion of Theorem 2 by induction.

**Proof of Theorem 3.** We first consider the case \( m=1 \). Choose \( q' < s < p \) in Theorem 1, by Theorem 1 and Lemma 4-6, we obtain
\[ ||T_\vec{b}(f)||_{L^p(\nu)} \leq ||M(T_\vec{b}(f))||_{L^p(\nu)} \leq C||(T_\vec{b})^#(f)||_{L^p(\nu)} \]
\[ \leq C||\vec{b}||_{BMO} \left(||M^k(f)||_{L^p(\nu)} + ||M^k(T(f))||_{L^p(\nu)} + ||M_s(f)||_{L^p(\nu)}\right) \]
\[ \leq C||\vec{b}||_{BMO} \left(||f||_{L^p(\nu)} + ||T(f)||_{L^p(\nu)} + ||f||_{L^p(\nu)}\right) \]
\[ \leq C||\vec{b}||_{BMO} \left(||f||_{L^p(\nu)} + ||f||_{L^p(\nu)}\right) \]
\[ \leq C||\vec{b}||_{BMO}||f||_{L^p(\nu)}. \]

When \( m \geq 2 \), we may get the conclusion of Theorem 3 by induction.

This completes the proof of Theorem 3.

**References**


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