Boundedness for multilinear commutator of singular integral operator with weighted Lipschitz functions

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ABSTRACT. In this paper, the boundedness for the multilinear commutators related to the singular integral operator with weighted Lipschitz functions is proved.

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1. Introduction

Let $b$ be a locally integrable function on $\mathbb{R}^n$ and $T$ be the Calderón-Zygmund operator. The commutator $[b, T]$ generated by $b$ and $T$ is defined by

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x).$$

In [6] and [17-19], the authors proved that the commutators and multilinear operators generated by the singular integral operators and $BMO$ functions are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Chanillo (see [4]) proved that the commutator $[b, I_\alpha]$ generated by $b \in BMO$ and the fractional integral operator $I_\alpha$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1 < p < q < \infty$ and $1/p - 1/q = \alpha/n$. Then Paluszyński (see [16]) showed that $b \in Lip_\beta(\mathbb{R}^n)$ (the homogeneous Lipschitz space) if and only if the commutator $[b, T]$ is bounded from $L^p$ to $L^q$, where $1 < p < r < \infty$, $0 < \beta < 1$ and $1/r = 1/p - (\beta + \alpha)/n$. Also Paluszyński (see [5], [10], [16]) obtain that $b \in Lip_\beta$ if and only if the commutator $[b, I_\alpha]$ is bounded from $L^p$ to $L^q$, where $1 < p < r < \infty$, $0 < \beta < 1$ and $1/r = 1/p - (\beta + \alpha)/n$ with $1/p > (\beta + \alpha)/n$.

On the other hand, in [1] and [9], the boundedness for the commutators generated by the singular integral operators and the weighted $BMO$ and Lipschitz functions on $L^p(\mathbb{R}^n)$ ($1 < p < \infty$) spaces are obtained. The purpose of this paper is to establish boundedness for the multilinear commutators related to the singular integral operator with general kernel (see [3] and [11]) and $b \in Lip_\beta(w)$ (the weighted Lipschitz space).

Definition 1.1. Let $T : S \to S'$ be a linear operator such that $T$ is bounded on $L^2(\mathbb{R}^n)$ and there exists a locally integrable function $K(x, y)$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$ such that

$$T(f)(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy$$

for every bounded and compactly supported function $f$, where $K$ satisfies: there is a sequence of positive constant numbers $\{C_k\}$ such that for any $k \geq 1$,

$$\int_{2|y-z|<|x-y|}(|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)|)dx \leq C,$$
and
\[
\left( \int_{2^k |z-y| \leq |x-y| < 2^{k+1} |z-y|} (|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)|)^q dy \right)^{1/q} \leq C_k (2^k |z-y|)^{-n/q'},
\]
where \(1 < q' < 2\) and \(1/q + 1/q' = 1\). Suppose \(b_j (j = 1, \ldots, m)\) are the fixed locally integrable functions on \(\mathbb{R}^n\). The multilinear commutator of the singular integral operator is defined by
\[
T_{\vec{b}}(f)(x) = \int_{\mathbb{R}^n} \prod_{j=1}^m (b_j(x) - b_j(y)) K(x,y) f(y) dy.
\]

Note that the classical Calderón-Zygmund singular integral operator satisfies Definition 1.1 with \(C_j = 2^{-j\delta}\) (see [8] and [18]).

Also note that when \(m = 1\), \(T_{\vec{b}}\) is just the commutator what we mentioned above. It is well known that multilinear operator are of great interest in harmonic analysis and have been widely studied by many authors (see [12-14], [17-19]). In [18], Pérez and Trujillo-Gonzalez prove a sharp estimate for the multilinear commutator. The purpose of this paper has two-folds, first, we establish a weighted Lipschitz estimate for the multilinear commutator related to the generalized singular integral operators, and second, we obtain the weighted \(L^p\)-norm inequality and the weighted estimate on the Triebel-Lizorkin space for the multilinear commutator by using the weighted Lipschitz estimate.

2. Notations and Results

Throughout this paper, we will use \(C\) to denote an absolute positive constant, which is independent of the main parameters and not necessarily the same at each occurrent. \(Q\) will denote a cube of \(\mathbb{R}^n\) with sides parallel to the axes. For any locally integrable function \(f\), the sharp maximal function of \(f\) is defined by
\[
f^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,
\]
where, and in what follows, \(f_Q = |Q|^{-1} \int_Q f(y) dx\). It is well-known that (see [8] and [20])
\[
f^\#(x) \approx \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - c| dy.
\]

For \(1 \leq p < \infty\) and \(0 \leq \eta < n\), let
\[
M_{\eta,p}(f)(x) = \sup_{Q \ni x} \left( \frac{1}{|Q|^{1-p\eta/n}} \int_Q |f(y)|^p dy \right)^{1/p},
\]
which is the Hardy-Littlewood maximal function when \(p = 1\) and \(\eta = 0\).

The \(A_p\) weight is defined by (see [8])
\[
A_p = \left\{ w : \sup_Q \left( \frac{1}{|Q|} \int_Q w(x) dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty \right\}, \quad 1 < p < \infty,
\]
\[
A_1 = \{ w > 0 : M(w)(x) \leq C w(x), a.e. \},
\]
and \(A_\infty = \cup_{p \geq 1} A_p\). We know that, for \(w \in A_1\), \(w\) satisfies the double condition, that is, for any cube \(Q\),
\[
w(2Q) \leq C w(Q).
\]
The $A(p, r)$ weight is defined by (see [15]), for $1 < p, r < \infty$,
$$A(p, r) = \left\{ w > 0 : \sup_Q \left( \frac{1}{|Q|} \int_Q w(x)^r dx \right)^{1/r} \left( \frac{1}{|Q|} \int_Q w(x)^{-p/(p-1)} dx \right)^{(p-1)/p} < \infty \right\}.$$ 

Given a weight function $w$ and $1 < p < \infty$, the weighted Lebesgue space $L^p(w)$ is the space of functions $f$ such that
$$\|f\|_{L^p(w)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

For a weight function $w$, $\beta > 0$ and $p > 1$, let $\hat{F}^{m, \beta}_p(w)$ be the weighted homogeneous Triebel-Lizorkin space (see [2]). For $0 < \beta < 1$, the weighted Lipschitz space $\text{Lip}_\beta(w)$ is the space of functions $f$ such that
$$\|f\|_{\text{Lip}_\beta(w)} = \sup_Q \frac{1}{w(Q)^{1+\beta/n}} \int_Q |f(y) - f_Q| dy < \infty.$$

Given some function $b_j \in \text{Lip}_\beta(w)$, $1 \leq j \leq m$, we denote by $C^m_j$ the family of all finite subsets $\sigma = (\sigma(1), \ldots, \sigma(j))$ of $\{1, \ldots, m\}$ of $j$ different elements and $\sigma(i) < \sigma(j)$ when $i < j$. For $\sigma \in C^m_j$, set $\sigma' = (1, \ldots, m) \setminus \sigma$. For $\tilde{b} = (b_1, \ldots, b_m)$ and $\sigma = (\sigma(1), \ldots, \sigma(j)) \in C^m_j$, set $\tilde{b}_\sigma = (b_{\sigma(1)}, \ldots, b_{\sigma(j)})$, $b_\sigma = \prod_{i=1}^j b_{\sigma(i)}$ and $||b_\sigma||_{\text{Lip}_\beta(w)} = ||b_{\sigma(1)}||_{\text{Lip}_\beta(w)} \cdots ||b_{\sigma(j)}||_{\text{Lip}_\beta(w)}$.

Now two theorems are stated out as following.

**Theorem 2.1.** Let $b_j \in \text{Lip}_\beta(w)$ for $1 \leq j \leq m$, $0 < \beta < 1$ and $w \in A_1$. Suppose the sequence $\{k^m C_k\} \in l^1$, $q' < p < \frac{n}{m \beta}$ and $\frac{1}{r} = \frac{1}{p} - \frac{m \beta}{n}$. Then $T_{b, \sigma}$ is bounded from $L^p(w)$ to $L^r(w^{1-m+(r-1)m \beta})$.

**Theorem 2.2.** Let $b_j \in \text{Lip}_\beta(w)$ for $1 \leq j \leq m$, $0 < \beta < 1$ and $w \in A_1$. Suppose the sequence $\{k^m C_k\} \in l^1$, $q' < p < \infty$. Then $T_{b, \sigma}$ is bounded from $L^p(w)$ to $\hat{F}^{m, \beta, \infty}_p(w^{1-m - \frac{m \beta}{r}})$.

### 3. Proofs of Theorems

In order to prove the theorems, the following lemmas are needed.

**Lemma 3.1.** (see [7], [9]) For $0 < \beta < 1$, $w \in A_1$, $b \in \text{Lip}_\beta(w)$ and $1 \leq p \leq \infty$, we have
$$||b||_{\text{Lip}_\beta(w)} \lesssim \sup_B w(B)^{-\beta} \left( w(B)^{-1} \int_B |b(x) - b_Q|^p w(x)^{-p} dx \right)^{1/p}.$$

**Lemma 3.2.** (see [7], [9]) For $0 < \beta < 1$, $w \in A_1$, $b \in \text{Lip}_\beta(w)$ and any cube $Q$, we have
$$\sup_{x \in Q} |b(x) - b_Q| \lesssim C ||b||_{\text{Lip}_\beta(w)} w(Q)^{1+\beta} |Q|^{-1}.$$

**Lemma 3.3.** (see [7], [9]) For $0 < \beta < 1$, $w \in A_1$, $b \in \text{Lip}_\beta(w)$, any cube $Q$ and $\tilde{x} \in Q$, we have
$$|b_{2Q} - b_Q| \lesssim C \nu w(\tilde{x}) 2^k |Q|^{\beta} ||b||_{\text{Lip}_\beta(w)}.$$

**Lemma 3.4.** (see [2]) For $0 < \beta < 1$, $w \in A_1$, $1 < p < \infty$ and $m > 0$, we have
$$||f||_{\hat{F}^{m, \beta, \infty}_p(w)} \approx \left( \sup_{|Q| \leq x} \left| \int_Q |f(x) - f_Q| dx \right|_{L^p(w)} \right)^{1/p} \\
\approx \left( \sup_{|Q| \leq x} \left| \int_Q |f(x) - C_0| dx \right|_{L^p(w)} \right)^{1/p}.$$
Lemma 3.5. (see [15]) Suppose that $1 \leq s < p < \eta$, $1/r = 1/p - \eta/n$ and $w \in A(p, r)$. Then
\[
\|M_{\eta,s}(f)\|_{L^r(w^r)} \leq C\|f\|_{L^p(w^p)}.
\]

Proof of Theorem 2.1. In order to prove the theorem, we will prove a sharp function estimate for the multilinear operator. We will prove that for any cube $Q$ and $q' < s < \infty$, there exists some constant $C_0$ such that
\[
\frac{1}{|Q|} \int_Q |T_\beta(f)(x) - C_0|dx \leq C\|\vec{\beta}\|_{Lip(\omega)}\|w(\vec{x})\|^{m+\frac{m\beta}{s}} (M_{m\beta,s}(f)\vec{x} + M_{m\beta,s}(T(f))\vec{x})
\]
Fix a cube $Q = Q(x_0, r_0)$ and $\vec{x} \in Q$, we write $f = f_1 + f_2$ with $f_1 = f\chi_{2Q}$, $f_2 = f\chi_{(2Q)\setminus Q}$. We first consider the Case $m = 1$. For $C_0 = T(((b_1)_1 b_1 f_1)(x_0)$, we write
\[
T_{b_1}(f)(x) = (b_1(x) - (b_1)_{2Q})T(f)(x) - T((b_1 - (b_1)_{2Q})f)(x).
\]
Then
\[
|T_{b_1}(f)(x) - C_0| \leq A(x) + B(x) + C(x),
\]
where
\[
A(x) = |(b_1(x) - (b_1)_{2Q})T(f)(x)|,
B(x) = |T(((b_1)_{2Q} - b_1)f_1)(x)|,
C(x) = |T(((b_1)_{2Q} - b_1)f_2)(x) - T(((b_1)_{2Q} - b_1)f_2)(x_0)|.
\]
For $A(x)$, by Hölder’s inequality and Lemma 3.2, we have
\[
\frac{1}{|Q|} \int_Q |A(x)|dx = \frac{1}{|Q|} \int_Q |b_1(x) - (b_1)_{2Q}||T(f)(x)|dx
\leq \frac{1}{|Q|} \left( \int_Q |b_1(x) - (b_1)_{2Q}|^s dx \right)^\frac{1}{s} \left( \int_Q |T(f)(x)|^r dx \right)^\frac{1}{r}
\leq \frac{1}{|2Q|} \sup_{x \in 2Q} |b_1(x) - (b_1)_{2Q}| |Q|^\frac{1}{r} \left( \int_Q |T(f)(x)|^r dx \right)^\frac{1}{r}
\leq C\frac{|b_1|_{Lip(\omega)}}{|Q|^\frac{1}{r}} \left( \int_Q (w')^{1+\frac{3}{r}} |Q|^{-1} |Q|^{\frac{1}{r}} |Q|^{\frac{1}{r}} \left( \frac{1}{|Q|^1} \int_Q |T(f)(x)|^r dx \right)^\frac{1}{r}
\leq C\|b_1\|_{Lip(\omega)} (w(2Q))^{1+\frac{3}{r}} M_{\beta,s}(T(f))(\vec{x})
\leq C\|b_1\|_{Lip(\omega)} (w(\vec{x}))^{1+\frac{3}{r}} M_{\beta,s}(T(f))(\vec{x}).
\]
For $B(x)$, by the type $(s,s)$ of $T$ and Lemma 3.2, we obtain
\[
\frac{1}{|Q|} \int_Q B(x)dx \leq C\frac{1}{|Q|} \left( \int_{R^n} |T(((b_1)_{Q} - b_1)f_1)(x)|^r dx \right)^\frac{1}{r} |Q|^{\frac{1}{r}}
\]
\[
\begin{align*}
&\leq C \frac{1}{|Q|} \left( \int_{\mathbb{R}^n} |(b_1(x) - b_1)|^p |f_1(x)|^{p'} \, dx \right)^{\frac{1}{p}} |Q|^{\frac{1}{p'}} \\
&\leq C \frac{1}{|Q|} \sup_{x \in 2Q} |b_1(x) - (b_1)_Q| \left( \int_{2Q} |f(x)|^p \, dx \right)^{\frac{1}{p}} |Q|^{\frac{1}{p'}} \\
&\leq C \frac{1}{|Q|} |b_1|_{L^p(\nu)} w(2Q)^{1 + \frac{\beta}{p}} |2Q|^{1 - \frac{\beta}{p}} \left( \frac{1}{|2Q|^{1 - \frac{\beta}{p}}} \int_{2Q} |f(x)|^p \, dx \right)^{\frac{1}{p}} |Q|^{\frac{1}{p'}} \\
&\leq C |b_1|_{L^p(\nu)} \left( \frac{w(2Q)}{|2Q|} \right)^{1 + \frac{\beta}{p}} M_{\beta,s}(f)(x) \\
&\leq C |b_1|_{L^p(\nu)} w(\bar{x})^{1 + \frac{\beta}{p}} M_{\beta,s}(f)(\bar{x}).
\end{align*}
\]

For \( C(x) \), recalling that \( s > q' \), taking \( 1 < p < \infty, 1 < t < s \) with \( 1/p + 1/q + 1/t = 1 \), by the Hölder’s inequality and Lemmas 3.1 and 3.3, we have, for \( x \in Q \),

\[
\begin{align*}
&T((b_1 - (b_1)_{2Q})f_2)(x) - T((b_1 - (b_1)_{2Q})f_2)(x_0)) \\
&\leq \int_{(2Q)^c} |b_1(y) - (b_1)_{2Q}| |f(y)||!K(x, y) - K(x_0, y)| dy \\
&\leq C \sum_{k=1}^{\infty} \left( \int_{2^k |x-x_0| \leq |y-x_0| < 2^{k+1}|x-x_0|} |K(x, y) - (x_0, y)|^p dy \right)^{1/q} \\
&\times \left( \int_{|y-x_0| < 2^{k+1}|x-x_0|} |b_1(y) - (b_1)_{2Q}|^p dy \right)^{1/p} \left( \int_{|y-x_0| < 2^{k+1}|x-x_0|} |f(y)|^t dy \right)^{1/t} \\
&\leq C \sum_{k=1}^{\infty} C_k \frac{1}{(2^k d)^{\frac{s}{p}}} \left( \int_{|y-x_0| < 2^{k+1}|x-x_0|} |b_1(y) - (b_1)_{2Q}|^p dy \right)^{1/p} \left( \int_{2^{k+1}Q} |f(y)|^t dy \right)^{1/t} \\
&\leq C \sum_{k=1}^{\infty} C_k \frac{1}{(2^k d)^{\frac{s}{p}}} \left( \int_{2^{k+1}Q} |b_1(y) - (b_1)_{2Q}|^p dy \right)^{1/p} \left( \int_{2^{k+1}Q} |f(y)|^t dy \right)^{1/t} \\
&+ C \sum_{k=1}^{\infty} C_k \frac{1}{(2^k d)^{\frac{s}{p}}} \left( \int_{2^{k+1}Q} |(b_1)_{2Q} - (b_1)_{2Q}|^p dy \right)^{1/p} \left( \int_{2^{k+1}Q} |f(y)|^t dy \right)^{1/t} \\
&\leq C \sum_{k=1}^{\infty} C_k \frac{1}{|2^{k+1}Q|^{\frac{s}{p}}} \sup_{y \in 2^{k+1}Q} |b_1(y) - (b_1)_{2Q}| |2^{k+1}Q|^{\frac{s}{p}} |2^{k+1}Q|^{\frac{s}{p} - \frac{\beta}{p}} \\
&\times \left( \frac{1}{|2^{k+1}Q|^{1 - \frac{\beta}{p}}} \int_{2^{k+1}Q} |f(y)|^t dy \right)^{1/s} + C \sum_{k=1}^{\infty} C_k \frac{1}{|2^{k+1}Q|^{\frac{s}{p}}} |(b_1)_{2Q} - (b_1)_{2Q}| \\
&\times |2^{k+1}Q|^{\frac{s}{p}} |2^{k+1}Q|^{\frac{s}{p} - \frac{\beta}{p}} \left( \frac{1}{|2^{k+1}Q|^{1 - \frac{\beta}{p}}} \int_{2^{k+1}Q} |f(y)|^t dy \right)^{1/s}.
\end{align*}
\]
Thus, recall that
\[ \int \frac{1}{Q} C(x) dx \leq C \| b_1 \|_{Lip(w)} w(\bar{x})^{1 + \frac{d}{Q}} M_{\beta,s}(f)(\bar{x}). \]

Now, we consider the Case \( m \geq 2 \). We have, for \( b = (b_1, \ldots, b_m) \),

\[
T_\beta(f)(x) = \int_{\mathbb{R}^n} \prod_{j=1}^m (b_j(x) - b_j(y)) K(x, y) f(y) dy \\
= \int_{\mathbb{R}^n} \prod_{j=1}^m [(b_j(x) - (b_j)_{2Q}) - (b_j(y) - (b_j)_{2Q})] K(x, y) f(y) dy \\
= \sum_{j=0}^m \sum_{\sigma \in C_j^m} (-1)^{m-j} \prod_{j=1}^m (b_j(x) - (b_j)_{2Q}) f(y),
\]

\[
\sigma \cdot \int_{\mathbb{R}^n} (b_j(y) - (b_j)_{2Q}) K(x, y) f(y) dy \\
+ (-1)^m \prod_{j=1}^m (b_j(x) - (b_j)_{2Q}) K(x, y) f(y) dy \\
+ \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} \prod_{j=1}^m (b_j(x) - (b_j)_{2Q}) f(y) dy \\
= \prod_{j=1}^m (b_j(x) - (b_j)_{2Q}) T(f)(x) + (-1)^m T(\prod_{j=1}^m (b_j - (b_j)_{2Q}) f)(x) \\
+ \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} (b_j(x) - (b_j)_{2Q}) f(y)
\]

thus, recall that \( C_0 = T(\prod_{j=1}^m (b_j - (b_j)_{2Q}) f_2)(x_0) \).
\[ |T_\tilde{b}(f)(x) - T(\prod_{j=1}^{m}(b_j - (b_j)_{2Q})f_2)(x_0)| \]

\[ \leq \prod_{j=1}^{m}(b_j(x) - (b_j)_{2Q})T(f)(x) + |T(\prod_{j=1}^{m}(b_j - (b_j)_{2Q})f_1)(x)| \]

\[ + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} ((b_j(x) - (b_j)_{2Q})\sigma T(b_j - (b_j)_{2Q})_{\sigma}(f)(x)| \]

\[ + |T(\prod_{j=1}^{m}(b_j - (b_j)_{2Q})f_2)(x) - T(\prod_{j=1}^{m}(b_j - (b_j)_{2Q})f_2)(x_0)| \]

\[ = I_1(x) + I_2(x) + I_3(x) + I_4(x). \]

where

\[ I_1(x) = \prod_{j=1}^{m}(b_j(x) - (b_j)_{2Q})T(f)(x), \]

\[ I_2(x) = |T(\prod_{j=1}^{m}(b_j - (b_j)_{2Q})f_1)(x)|, \]

\[ I_3(x) = |\sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} ((b_j(x) - (b_j)_{2Q})\sigma T(b_j - (b_j)_{2Q})_{\sigma}(f)(x)|, \]

\[ I_4(x) = |T(\prod_{j=1}^{m}(b_j - (b_j)_{2Q})f_2)(x) - T(\prod_{j=1}^{m}(b_j - (b_j)_{2Q})f_2)(x_0)|. \]

For \( I_1(x) \), by Hölder’s inequality with exponent \( \frac{1}{r_1} + \cdots + \frac{1}{r_m} + \frac{1}{s} = 1 \) and Lemma 3.2, we get

\[ \frac{1}{|Q|} \int_Q I_1(x)dx \leq C \frac{1}{|Q|} \int_Q \prod_{j=1}^{m}(b_j(x) - (b_j)_{2Q})||T(f)(x)||dx \]

\[ \leq C \frac{1}{|Q|} \prod_{j=1}^{m} \left( \int_Q |b_j(x) - (b_j)_{2Q}|^{r_j}dx \right)^{1/r_j} \left( \int_Q |T(f)(x)|^sdx \right)^{1/s} \]

\[ \leq C \frac{1}{|Q|} \prod_{j=1}^{m} \left( \sup_{x \in Q} |b_j(x) - (b_j)_{2Q}|^{1/r_j} \right) \left( \int_Q |T(f)(x)|^sdx \right)^{1/s} \]

\[ \leq C \frac{1}{|Q|} \prod_{j=1}^{m} \left( ||b_j||_{Lip_{\beta}(w)} w(Q)^{1+\frac{m}{s}|Q|-1})|Q|^{1-\frac{1}{s}+\frac{1}{r_m}+\frac{m}{s}} \right) \left( \frac{1}{|Q|^{1-\frac{1}{s}}} \int_Q |T(f)(x)|^sdx \right)^{1/s} \]

\[ \leq C ||\tilde{b}||_{Lip_{\beta}(w)} w(Q)^{m+\frac{m}{s}} |Q|^{1-\frac{1}{s}+\frac{1}{r_m}+\frac{m}{s}} M_{m,\beta,s}(T(f))(\tilde{x}) \]

\[ \leq C ||\tilde{b}||_{Lip_{\beta}(w)} \left( \frac{w(Q)}{|Q|} \right)^{m+\frac{m}{s}} M_{m,\beta,s}(T(f))(\tilde{x}) \]

\[ \leq C ||\tilde{b}||_{Lip_{\beta}(w)} w(\tilde{x})^{m+\frac{m}{s}} M_{m,\beta,s}(T(f))(\tilde{x}). \]
For $I_2(x)$, similar to $B(x)$, using the boundness of $T$ and Lemma 3.2, we get, for $1 < t < s$,

\[
\frac{1}{|Q|} \int_Q I_2(x) \, dx \leq C \frac{1}{|Q|} \left( \int_{\mathbb{R}^n} \left| T \left( \prod_{j=1}^m (b_j - (b_j)_{2Q}) f_1 \right) \right| \, dx \right)^{\frac{r}{r'}} |Q|^{\frac{r}{r'}} \]

\[
\leq C \frac{1}{|Q|} \left( \int_{\mathbb{R}^n} \left| \prod_{j=1}^m (b_j(x) - (b_j)_{2Q}) f_1(x) \right| \, dx \right)^{\frac{r}{r'}} |Q|^{\frac{r}{r'}} \]

\[
\leq C \frac{1}{|Q|} \left( \int_{Q} \left| \prod_{j=1}^m (b_j(x) - (b_j)_{2Q}) \right| |f(x)| \, dx \right)^{\frac{r}{r'}} |Q|^{\frac{r}{r'}} \]

\[
\leq C \frac{1}{|Q|} \left( \prod_{j=1}^m \| b_j \|_{Lip(w(2Q))} |Q| \right)^{\frac{r}{r'}} |Q|^{\frac{r}{r'}} \]

\[
\leq C \| \tilde{b} \|_{Lip(w(2Q))} \left( \frac{w(2Q)}{|2Q|} \right)^{\frac{m}{n}} \left( \frac{1}{|2Q|^{1 - \frac{m}{n}}} \int_{2Q} |f(x)| \, dx \right)^{\frac{r}{r'}} \]

For $I_3(x)$, by Hölder’s inequality and Lemma 3.2, we get

\[
\frac{1}{|Q|} \int_Q I_3(x) \, dx \leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_{j}^m} \frac{1}{|2Q|} \int_{2Q} |(b_j(x) - (b_j)_{2Q})_{\sigma} |T((b_j - (b_j)_{2Q})_\sigma f)(x)| \, dx \]

\[
\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_{j}^m} \frac{1}{|2Q|} \left( \int_{2Q} |(b_j(x) - (b_j)_{2Q})_{\sigma} | \, dx \right)^{\frac{r}{r'}} \]

\[
\times \left( \int_{2Q} \left| T((b_j - (b_j)_{2Q})_\sigma f)(x) \right| \, dx \right)^{\frac{r'}{r'}} \]

\[
\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_{j}^m} \frac{1}{|2Q|} \sup_{x \in 2Q} |(b_j(x) - (b_j)_{2Q})_{\sigma} | |2Q|^{\frac{r}{r'}} \]

\[
\times \left( \int_{2Q} |(b_j(x) - (b_j)_{2Q})_{\sigma} | |T(f)(x)| \, dx \right)^{\frac{r'}{r'}} \]

\[
\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_{j}^m} \frac{1}{|2Q|} \sup_{x \in 2Q} |(b_j(x) - (b_j)_{2Q})_{\sigma} | |2Q|^{\frac{r}{r'}} \]

\[
\times \sup_{x \in 2Q} |(b_j(x) - (b_j)_{2Q})_{\sigma} | \left( \int_{2Q} |T(f)(x)| \, dx \right)^{\frac{r}{r'}} \]

\[
\leq C \frac{1}{|2Q|} \| b_{\sigma} \|_{Lip(w(2Q)^{j+\frac{r'}{r} - j} |2Q|-j} \| b_{\sigma} \|_{Lip(w(2Q) \sigma^{-1}(j-\frac{m-1}{n})}

\[
\times |2Q|^{\frac{r}{r'} + \frac{m}{n} - (m-j)} \left( \frac{1}{|2Q|^{1 - \frac{m}{n}}} \int_{2Q} |T(f)(x)| \, dx \right)^{\frac{r}{r'}} \]
\[
\begin{align*}
&\leq C\|\tilde{b}\|_{Lip_b(w)} \left( \frac{w(2Q)}{|2Q|} \right)^{m+\frac{m_a}{r}} M_{m_b,s}(T(f))(\tilde{x}) \\
&\leq C\|\tilde{b}\|_{Lip_b(w)} w(\tilde{x})^{m+\frac{m_a}{r}} M_{m_b,s}(T(f))(\tilde{x}).
\end{align*}
\]

For \( I_4(x) \), similar to the proof of \( C(x) \) in the Case \( m = 1 \), for \( 1 < p < \infty, 1 < t < s \) with \( 1/p + 1/q + 1/t = 1 \), we have

\[
|T(\prod_{j=1}^m ((b_j(y) - (b_j)_{2Q})f_2(x)) - T(\prod_{j=1}^m ((b_j - (b_j)_{2Q})f_2(x_0))|
\]

\[
\leq \int_{(2Q)^c} \left| \prod_{j=1}^m (b_j(y) - (b_j)_{2Q}) \right| |f(y)||K(x, y) - K(x_0, y)| dy
\]

\[
\leq C \sum_{k=1}^\infty \left( \int_{2^k |x-x_0| \leq |y-x_0| < 2^{k+1} |x-x_0|} |K(x, y) - K(x_0, y)|^q dy \right)^{1/q}
\]

\[
\times \left( \int_{|y-x_0| < 2^{k+1} |x-x_0|} \prod_{j=1}^m b_1(y) - (b_j)_{2Q} |^p dy \right)^{1/p}
\]

\[
\leq C \sum_{k=1}^\infty C_k \frac{1}{(2d)^{\frac{q}{p}}} \left( \int_{|y-x_0| < 2^{k+1} |x-x_0|} \prod_{j=1}^m b_1(y) - (b_j)_{2Q} |^p dy \right)^{1/p}
\]

\[
\times \left( \int_{|y-x_0| < 2^{k+1} |x-x_0|} |f(y)|^q dy \right)^{1/q}
\]

\[
\leq C \sum_{k=1}^\infty C_k \frac{1}{(2d)^{\frac{q}{p}}} \left( \int_{2^{k+1} Q} \prod_{j=1}^m (b_j(y) - (b_j)_{2^{k+1} Q} |^p dy \right)^{1/p}
\]

\[
\times \left( \int_{2^{k+1} Q} |f(y)|^q dy \right)^{1/q}
\]

\[
\leq C \sum_{k=1}^\infty C_k \frac{1}{|2^k Q|^\frac{q}{p}} \prod_{j=1}^m \sup_{x \in 2^{k+1} Q} |b_j(y) - (b_j)_{2^{k+1} Q}|^\frac{1}{2^{k+1} Q} \left| 2^{k+1} Q \right|^\frac{1}{2^{k+1} Q}^\frac{m_a}{r}
\]

\[
\times \left( \int_{2^{k+1} Q} |f(y)|^q dy \right)^{1/s} + C \sum_{k=1}^\infty C_k \frac{1}{|2^k Q|^\frac{q}{p}} \prod_{j=1}^m |(b_j)_{2^{k+1} Q} - (b_j)_{2Q}|
\]

\[
\times \left| 2^{k+1} Q \right|^\frac{1}{2^{k+1} Q} \left| 2^{k+1} Q \right|^\frac{m_a}{r} \left( \frac{1}{\left| 2^{k+1} Q \right|^\frac{1}{2^{k+1} Q} \left| 2^{k+1} Q \right|^\frac{m_a}{r}} \int_{2^{k+1} Q} |f(y)|^q dy \right)^{1/s}
\]

\[
\leq C \sum_{k=1}^\infty C_k \|\tilde{b}\|_{Lip_b(w)} w(2^{k+1} Q)^{m+\frac{m_a}{r}} |2^{k+1} Q|^{-m} |2^{k+1} Q|^{-\frac{m_a}{r}} M_{m_b,s}(f)(\tilde{x})
\]

\[
+ C \sum_{k=1}^\infty C_k w(\tilde{x})^{m+\frac{m_a}{r}} \|\tilde{b}\|_{Lip_b(w)} |2^{k+1} Q|^{-m} |2^{k+1} Q|^{-\frac{m_a}{r}} M_{m_b,s}(f)(\tilde{x})
\]
\[
\begin{align*}
\leq & \quad C|\tilde{b}|_{L_{p,q}(u)} \sum_{k=1}^{\infty} k^{m} C_k \left( \frac{w(2^{k+1}Q)}{2^{k+1}Q} \right)^m M_{m,s}(f)(\tilde{x}) \\
+ & \quad C|\tilde{b}|_{L_{p,q}(u)} \sum_{k=1}^{\infty} k^{m} C_k w(\tilde{x})^m \left( \frac{w(2^{k+1}Q)}{2^{k+1}Q} \right)^{m/s} M_{m,s}(f)(\tilde{x}) \\
\leq & \quad C|\tilde{b}|_{L_{p,q}(u)} w(\tilde{x})^m + \frac{m^a}{n} M_{m,s}(f)(\tilde{x}),
\end{align*}
\]

thus, we get
\[
\frac{1}{|Q|} \int_{Q} I_4(x) dx \leq C|\tilde{b}|_{L_{p,q}(w)} w(\tilde{x})^{m + \frac{m^a}{n}} M_{m,s}(f)(\tilde{x}).
\]

Combining all the estimates above, we get
\[
\frac{1}{|Q|} \int_{Q} |T_{f}^{\ast}(f)(x) - C_0| dx \leq C|\tilde{b}|_{L_{p,q}(w)} w(\tilde{x})^{m + \frac{m^a}{n}} (M_{m,s}(f)(\tilde{x}) + M_{m,s}(T(f))(\tilde{x}))
\]
and
\[
T_{f}^{\ast}(f)\#(\tilde{x}) \leq C|\tilde{b}|_{L_{p,q}(w)} w(\tilde{x})^{m + \frac{m^a}{n}} (M_{m,s}(f)(\tilde{x}) + M_{m,s}(T(f))(\tilde{x})).
\]

Now, choose \(q' < s < r\), by Lemma 3.5, we have
\[
\begin{align*}
||T_{f}^{\ast}(f)||_{L_{q'}(w^{1-m+(q-1)\frac{m^a}{n}})} & \leq C||M(T_{f}^{\ast}(f))||_{L_{q}(w^{1-m+(q-1)\frac{m^a}{n}})} \\
& \leq C||T_{f}^{\ast}(f)||_{L_{q}(w^{1-m+(q-1)\frac{m^a}{n}})} \\
& \leq C|\tilde{b}|_{L_{p,q}(w)} (||w^{m+\beta/n} M_{m,s}(f)||_{L_{q}(w^{1-m+(q-1)\frac{m^a}{n}})} + ||w^{m+\beta/n} M_{m,s}(T(f))||_{L_{q}(w^{1-m+(q-1)\frac{m^a}{n}})}) \\
& \leq C|\tilde{b}|_{L_{p,q}(w)} (||M_{m,s}(f)||_{L_{q}(w^{1-m+(q-1)\frac{m^a}{n}})} + ||M_{m,s}(T(f))||_{L_{q}(w^{1-m+(q-1)\frac{m^a}{n}})}) \\
& \leq C|\tilde{b}|_{L_{p,q}(w)} (||f||_{L_{p}(w)} + ||T(f)||_{L_{p}(w)}) \\
& \leq C|\tilde{b}|_{L_{p,q}(w)} ||f||_{L_{p}(w)}.
\end{align*}
\]

This completes the proof of Theorem 2.1. \(\square\)

**Proof of Theorem 2.2.** Similar to Theorem 2.1, for any \(q' < s < \infty\) and cube \(Q\), there exists some constant \(C_Q\) such that for \(f \in L^p(w)\) and \(\tilde{x} \in Q\),
\[
|Q|^{-1 - \frac{m^a}{n}} \int_{Q} |T_{f}^{\ast}(f)(x) - C_0| dx \leq C|\tilde{b}|_{L_{p,q}(w)} w(\tilde{x})^{m + \frac{m^a}{n}} (M_{s}(f)(\tilde{x}) + M_{s}(T(f))(\tilde{x})).
\]

Further, we have
\[
\sup_{Q \ni \tilde{x} \in C} \inf_{Q} |Q|^{-1 - \frac{m^a}{n}} \int_{Q} |T_{f}^{\ast}(f)(x)/(x) - c| dx \leq C|\tilde{b}|_{L_{p,q}(w)} w(\tilde{x})^{m + \frac{m^a}{n}} (M_{s}(f)(\tilde{x}) + M_{s}(T(f))(\tilde{x})).
\]

Choose \(q' < s < p\) and by Lemma 3.4, we obtain
\[
\begin{align*}
||T_{f}^{\ast}(f)||_{L_{p,q}(w^{1-m-\frac{m^a}{n}})} & \approx \left( \sup_{\tilde{x} \in C} |Q|^{-1 - \frac{m^a}{n}} \int_{Q} |T_{f}^{\ast}(f)(x) - c| dx \right)_{L_{p}(w^{1-m-\frac{m^a}{n}})} \\
& \leq C|\tilde{b}|_{L_{p,q}(w)} (||w^{m+\beta/n} M_{s}(f)||_{L_{p}(w^{1-m-\frac{m^a}{n}})} + ||w^{m+\beta/n} M_{s}(T(f))||_{L_{p}(w^{1-m-\frac{m^a}{n}})}) \\
& \leq C|\tilde{b}|_{L_{p,q}(w)} (||M_{s}(f)||_{L_{p}(w)} + ||M_{s}(T(f))||_{L_{p}(w)}) \\
& \leq C|\tilde{b}|_{L_{p,q}(w)} (||f||_{L_{p}(w)} + ||T(f)||_{L_{p}(w)}) \\
& \leq C|\tilde{b}|_{L_{p,q}(w)} ||f||_{L_{p}(w)}.
\end{align*}
\]
This completes the proof of Theorem 2.2.

References