Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 40(1), 2013, Pages 84–94
ISSN: 1223-6934

Boundedness for multilinear commutator of singular integral operator with weighted Lipschitz functions

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Abstract. In this paper, the boundedness for the multilinear commutators related to the singular integral operator with weighted Lipschitz functions is proved.

2010 Mathematics Subject Classification. Primary 42B20; Secondary 42B25.
Key words and phrases. Multilinear commutator; Singular integral operator; Weighted Lipschitz function; Triebel-Lizorkin space.

1. Introduction

Let b be a locally integrable function on $\mathbb{R}^n$ and T be the Calderón-Zygmund operator. The commutator $[b, T]$ generated by b and T is defined by

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x).$$

In [6] and [17-19], the authors proved that the commutators and multilinear operators generated by the singular integral operators and BMO functions are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Chanillo (see [4]) proved that the commutator $[b, I_\alpha]$ generated by $b \in BMO$ and the fractional integral operator I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1 < p < q < \infty$ and $1/p - 1/q = \alpha/n$. Then Paluszyński (see [16]) showed that $b \in Lip_\beta(\mathbb{R}^n)$(the homogeneous Lipschitz space) if and only if the commutator $[b, T]$ is bounded from L^p to L^q, where $1 < p < q < \infty$, $0 < \beta < 1$ and $1/q = 1/p - \beta/n$. Also Paluszyński (see [5], [10], [16]) obtain that $b \in Lip_\beta$ if and only if the commutator $[b, I_\alpha]$ is bounded from L^p to L^r, where $1 < p < r < \infty$, $0 < \beta < 1$ and $1/r = 1/p - (\beta + \alpha)/n$ with $1/p > (\beta + \alpha)/n$.

On the other hand, in [1] and [9], the boundedness for the commutators generated by the singular integral operators and the weighted BMO and Lipschitz functions on $L^p(\mathbb{R}^n)(1 < p < \infty)$ spaces are obtained. The purpose of this paper is to establish boundedness for the multilinear commutators related to the singular integral operator with general kernel (see [3] and [11]) and $b \in Lip_\beta(w)$(the weighted Lipschitz space).

Definition 1.1. Let $T : S \to S'$ be a linear operator such that T is bounded on $L^2(\mathbb{R}^n)$ and there exists a locally integrable function $K(x, y)$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$ such that

$$T(f)(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy$$

for every bounded and compactly supported function f, where K satisfies: there is a sequence of positive constant numbers $\{C_k\}$ such that for any $k \geq 1$,

$$\int_{2|y-z|<|x-y|} (|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)|)dx \leq C,$$

Received January 3, 2012; Revised November 22, 2012.

and

$$\left(\int_{2^k|z-y|\le|x-y|<2^{k+1}|z-y|}(|K(x,y)-K(x,z)|+|K(y,x)-K(z,x)|)^q dy\right)^{1/q}$$
$$\le C_k(2^k|z-y|)^{-n/q'},$$

where $1<q'<2$ and $1/q+1/q'=1$. Suppose b_j $(j=1,\cdots,m)$ are the fixed locally integrable functions on R^n. The *multilinear commutator* of the singular integral operator is defined by

$$T_{\vec{b}}(f)(x)=\int_{R^n}\prod_{j=1}^m(b_j(x)-b_j(y))K(x,y)f(y)dy.$$

Note that the classical Calderón-Zygmund singular integral operator satisfies Definition 1.1 with $C_j=2^{-j\delta}$ (see [8] and [18]).

Also note that when $m=1$, $T_{\vec{b}}$ is just the commutator what we mentioned above. It is well known that multilinear operator are of great interest in harmonic analysis and have been widely studied by many authors (see [12-14], [17-19]). In [18], Pérez and Trujillo-Gonzalez prove a sharp estimate for the multilinear commutator. The purpose of this paper has two-folds, first, we establish a weighted Lipschitz estimate for the multilinear commutator related to the generalized singular integral operators, and second, we obtain the weighted L^p-norm inequality and the weighted estimate on the Triebel-Lizorkin space for the multilinear commutator by using the weighted Lipschitz estimate.

2. Notations and Results

Throughout this paper, we will use C to denote an absolute positive constant, which is independent of the main parameters and not necessarily the same at each occurrent. Q will denote a cube of $\mathbb{R}^n$ with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$f^{\#}(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)-f_Q|dy,$$

where, and in what follows, $f_Q=|Q|^{-1}\int_Q f(x)dx$. It is well-known that (see [8] and [20])

$$f^{\#}(x)\approx\sup_{Q\ni x}\inf_{c\in\mathcal{C}}\frac{1}{|Q|}\int_Q|f(y)-c|dy.$$

For $1\le p<\infty$ and $0\le\eta<n$, let

$$M_{\eta,p}(f)(x)=\sup_{Q\ni x}\left(\frac{1}{|Q|^{1-p\eta/n}}\int_Q|f(y)|^p dy\right)^{1/p},$$

which is the Hardy-Littlewood maximal function when $p=1$ and $\eta=0$.

The A_p weight is defined by (see [8])

$$A_p=\left\{w:\sup_Q\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1}<\infty\right\},\quad 1<p<\infty,$$

$$A_1=\{w>0:M(w)(x)\le Cw(x),a.e.\},$$

and $A_\infty=\cup_{p\ge1}A_p$. We know that, for $w\in A_1$, w satisfies the double condition, that is, for any cube Q,

$$w(2Q)\le Cw(Q).$$

The $A(p,r)$ weight is defined by (see [15]), for $1 < p, r < \infty$,

$$A(p,r) = \left\{ w > 0 : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)^r dx \right)^{1/r} \left(\frac{1}{|Q|} \int_Q w(x)^{-p/(p-1)} dx \right)^{(p-1)/p} < \infty \right\}.$$

Given a weight function w and $1 < p < \infty$, the weighted Lebesgue space $L^p(w)$ is the space of functions f such that

$$||f||_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

For a weight function w, $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta,\infty}(w)$ be the weighted homogeneous Triebel-Lizorkin space (see [2]). For $0 < \beta < 1$, the weighted Lipschitz space $Lip_\beta(w)$ is the space of functions f such that

$$||f||_{Lip_\beta(w)} = \sup_Q \frac{1}{w(Q)^{1+\beta/n}} \int_Q |f(y) - f_Q| dy < \infty.$$

Given some function $b_j \in Lip_\beta(w), 1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements and $\sigma(i) < \sigma(j)$ when $i < j$. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \backslash \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = \prod_{i=1}^{j} b_{\sigma(i)}$ and $||\vec{b_\sigma}||_{Lip_\beta(\omega)} = ||b_{\sigma(1)}||_{Lip_\beta(w)} \cdots ||b_{\sigma(i)}||_{Lip_\beta(w)}$.

Now two theorems are stated out as following.

Theorem 2.1. *Let $b_j \in Lip_\beta(w)$ for $1 \le j \le m$, $0 < \beta < 1$ and $w \in A_1$. Suppose the sequence $\{k^m C_k\} \in l^1$, $q' < p < \frac{n}{m\beta}$ and $\frac{1}{r} = \frac{1}{p} - \frac{m\beta}{n}$. Then $T_{\vec{b}}$ is bounded from $L^p(w)$ to $L^r(w^{1-m+(r-1)\frac{m\beta}{n}})$.*

Theorem 2.2. *Let $b_j \in Lip_\beta(w)$ for $1 \le j \le m$, $0 < \beta < 1$ and $w \in A_1$. Suppose the sequence $\{k^m C_k\} \in l^1$, $q' < p < \infty$. Then $T_{\vec{b}}$ is bounded from $L^p(w)$ to $\tilde{F}_p^{m\beta,\infty}(w^{1-m-\frac{m\beta}{n}})$.*

3. Proofs of Theorems

In order to prove the theorems, the following lemmas are needed.

Lemma 3.1. (see [7], [9]) *For $0 < \beta < 1$, $w \in A_1$, $b \in Lip_\beta(w)$ and $1 \le p \le \infty$, we have*

$$||b||_{Lip_\beta(w)} \approx \sup_B w(Q)^{-\beta} \left(w(Q)^{-1} \int_Q |b(x) - b_Q|^p w(x)^{1-p} dx \right)^{1/p}.$$

Lemma 3.2. (see [7], [9]) *For $0 < \beta < 1$, $w \in A_1$, $b \in Lip_\beta(w)$ and any cube Q, we have*

$$\sup_{x \in Q} |b(x) - b_Q| \le C||b||_{Lip_\beta(w)} w(Q)^{1+\beta} |Q|^{-1}.$$

Lemma 3.3. (see [7], [9]) *For $0 < \beta < 1$, $w \in A_1$, $b \in Lip_\beta(w)$, any cube Q and $\tilde{x} \in Q$, we have*

$$|b_{2^k Q} - b_Q| \le Ckw(\tilde{x}) w(2^k Q)^\beta ||b||_{Lip_\beta(w)}.$$

Lemma 3.4. (see [2]) *For $0 < \beta < 1$, $w \in A_1$, $1 < p < \infty$ and $m > 0$, we have*

$$\begin{aligned} ||f||_{\tilde{F}_p^{m\beta,\infty}(w)} &\approx \left|\left| \sup_{Q \ni \tilde{x}} |Q|^{-1-m\beta} \int_Q |f(x) - f_Q| dx \right|\right|_{L^p(w)} \\ &\approx \left|\left| \sup_{Q \ni \tilde{x}} \inf_{C_0 \in C} |Q|^{-1-m\beta} \int_Q |f(x) - C_0| dx \right|\right|_{L^p(w)}. \end{aligned}$$

Lemma 3.5. (see [15]) *Suppose that* $1 \le s < p < \eta$, $1/r = 1/p - \eta/n$ *and* $w \in A(p,r)$. *Then*

$$||M_{\eta,s}(f)||_{L^r(w^r)} \le C||f||_{L^p(w^p)}.$$

Proof of Theorem 2.1. In order to prove the theorem, we will prove a sharp function estimate for the multilinear operator. We will prove that for any cube Q and $q' < s < \infty$, there exists some constant C_0 such that

$$\frac{1}{|Q|}\int_Q |T_{\vec{b}}(f)(x) - C_0|dx \le C||\vec{b}||_{Lip_\beta(w)} w(\tilde{x})^{m+\frac{m\beta}{n}}(M_{m\beta,s}(f)(\tilde{x}) + M_{m\beta,s}(T(f))(\tilde{x})).$$

Fix a cube $Q = Q(x_0, r_0)$ and $\tilde{x} \in Q$, we write $f = f_1 + f_2$ with $f_1 = f\chi_{2Q}$, $f_2 = f\chi_{(2Q)^c}$.

We first consider the **Case** $m = 1$. For $C_0 = T(((b_1)_{2Q} - b_1)f_2)(x_0)$, we write

$$T_{b_1}(f)(x) = (b_1(x) - (b_1)_{2Q})T(f)(x) - T((b_1 - (b_1)_{2Q})f)(x).$$

Then

$$|T_{b_1}(f)(x) - C_0| \le A(x) + B(x) + C(x),$$

where

$$\begin{aligned} A(x) &= |(b_1(x) - (b_1)_{2Q})T(f)(x)|, \\ B(x) &= |T(((b_1)_{2Q} - b_1)f_1)(x)|, \\ C(x) &= |T(((b_1)_{2Q} - b_1)f_2)(x) - T(((b_1)_{2Q} - b_1)f_2)(x_0)|. \end{aligned}$$

For $A(x)$, by Hölder's inequality and Lemma 3.2, we have

$$\begin{aligned} \frac{1}{|Q|}\int_Q |A(x)|dx &= \frac{1}{|Q|}\int_Q |b_1(x) - (b_1)_{2Q}||T(f)(x)|dx \\ &\le \frac{1}{|Q|}\left(\int_Q |b_1(x) - (b_1)_{2Q}|^{s'}dx\right)^{\frac{1}{s'}}\left(\int_Q |T(f)(x)|^s dx\right)^{\frac{1}{s}} \\ &\le \frac{1}{|2Q|}\sup_{x\in 2Q}|b_1(x) - (b_1)_{2Q}||Q|^{\frac{1}{s'}}\left(\int_Q |T(f)(x)|^s dx\right)^{\frac{1}{s}} \\ &\le \frac{C}{|Q|}||b_1||_{Lip_\beta(w)} w(2Q)^{1+\frac{\beta}{n}}|Q|^{-1}|Q|^{\frac{1}{s'}}|Q|^{\frac{1}{s}-\frac{\beta}{n}}\left(\frac{1}{|Q|^{1-\frac{s\beta}{n}}}\int_Q |T(f)(x)|^r dx\right)^{\frac{1}{s}} \\ &\le C||b_1||_{Lip_\beta(w)}\left(\frac{w(2Q)}{|2Q|}\right)^{1+\frac{\beta}{n}} M_{\beta,s}(T(f))(\tilde{x}) \\ &\le C||b_1||_{Lip_\beta(w)} w(\tilde{x})^{1+\frac{\beta}{n}} M_{\beta,s}(T(f))(\tilde{x}). \end{aligned}$$

For $B(x)$, by the type (s,s) of T and Lemma 3.2, we obtain

$$\frac{1}{|Q|}\int_Q B(x)dx \le C\frac{1}{|Q|}\left(\int_{R^n} |T(((b_1)_Q - b_1)f_1)(x)|^s dx\right)^{\frac{1}{r}}|Q|^{\frac{1}{s'}}$$

$$
\begin{aligned}
&\le C\frac{1}{|Q|}\left(\int_{\mathbb{R}^n}|(b_1)_Q-b_1(x)|^r|f_1(x)|^r dx\right)^{\frac{1}{s}}|Q|^{\frac{1}{s'}}\\
&\le C\frac{1}{|Q|}\sup_{x\in 2Q}|b_1(x)-(b_1)_Q|\left(\int_{2Q}|f(x)|^s dx\right)^{\frac{1}{s}}|Q|^{\frac{1}{s'}}\\
&\le C\frac{1}{|Q|}||b_1||_{Lip_\beta(w)}w(2Q)^{1+\frac{\beta}{n}}|2Q|^{-1}|2Q|^{\frac{1}{s}-\frac{\beta}{n}}\left(\frac{1}{|2Q|^{1-\frac{s\beta}{n}}}\int_{2Q}|f(x)|^s dx\right)^{\frac{1}{s}}|Q|^{\frac{1}{s'}}\\
&\le C||b_1||_{Lip_\beta(w)}\left(\frac{w(2Q)}{|2Q|}\right)^{1+\frac{\beta}{n}}M_{\beta,s}(f)(x)\\
&\le C||b_1||_{Lip_\beta(w)}w(\tilde{x})^{1+\frac{\beta}{n}}M_{\beta,s}(f)(\tilde{x}).
\end{aligned}
$$

For $C(x)$, recalling that $s>q'$, taking $1<p<\infty, 1<t<s$ with $1/p+1/q+1/t=1$, by the Hölder's inequality and Lemmas 3.1 and 3.3, we have, for $x\in Q$,

$$
\begin{aligned}
&|T((b_1-(b_1)_{2Q})f_2)(x)-T((b_1-(b_1)_{2Q})f_2)(x_0)|\\
&\le \int_{(2Q)^c}|b_1(y)-(b_1)_{2Q}||f(y)||K(x,y)-K(x_0,y)|dy\\
&\le C\sum_{k=1}^{\infty}\left(\int_{2^k|x-x_0|\le|y-x_0|<2^{k+1}|x-x_0|}|K(x,y)-K(x_0,y)|^q dy\right)^{1/q}\\
&\times \left(\int_{|y-x_0|<2^{k+1}|x-x_0|}|b_1(y)-(b_1)_{2Q}|^p dy\right)^{1/p}\left(\int_{|y-x_0|<2^{k+1}|x-x_0|}|f(y)|^t dy\right)^{1/t}\\
&\le C\sum_{k=1}^{\infty}C_k\frac{1}{(2^k d)^{\frac{n}{q'}}}\left(\int_{|y-x_0|<2^{k+1}|x-x_0|}|b_1(y)-(b_1)_{2Q}|^p dy\right)^{1/p}\\
&\times \left(\int_{|y-x_0|<2^{k+1}|x-x_0|}|f(y)|^s dy\right)^{1/s}\\
&\le C\sum_{k=1}^{\infty}C_k\frac{1}{(2^k d)^{\frac{n}{q'}}}\left(\int_{2^{k+1}Q}|b_1(y)-(b_1)_{2Q}|^p dy\right)^{1/p}\left(\int_{2^{k+1}Q}|f(y)|^t dy\right)^{1/t}\\
&\le C\sum_{k=1}^{\infty}C_k\frac{1}{(2^k d)^{\frac{n}{q'}}}\left(\int_{2^{k+1}Q}|b_1(y)-(b_1)_{2^{k+1}Q}|^p dy\right)^{1/p}\left(\int_{2^{k+1}Q}|f(y)|^t dy\right)^{1/t}\\
&+ C\sum_{k=1}^{\infty}C_k\frac{1}{(2^k d)^{\frac{n}{q'}}}\left(\int_{2^{k+1}Q}|(b_1)_{2^{k+1}Q}-(b_1)_{2Q}|^p dy\right)^{1/p}\left(\int_{2^{k+1}Q}|f(y)|^t dy\right)^{1/t}\\
&\le C\sum_{k=1}^{\infty}C_k\frac{1}{|2^{k+1}Q|^{\frac{1}{q'}}}\sup_{y\in 2^{k+1}Q}|b_1(y)-(b_1)_{2^{k+1}Q}||2^{k+1}Q|^{\frac{1}{p}}|2^{k+1}Q|^{\frac{1}{s}-\frac{\beta}{n}}\\
&\times \left(\frac{1}{|2^{k+1}Q|^{1-\frac{s\beta}{n}}}\int_{2^{k+1}Q}|f(y)|^s dy\right)^{1/s}+C\sum_{k=1}^{\infty}C_k\frac{1}{|2^{k+1}Q|^{\frac{1}{q'}}}|(b_1)_{2^{k+1}Q}-(b_1)_{2Q}|\\
&\times \ |2^{k+1}Q|^{\frac{1}{p}}|2^{k+1}Q|^{\frac{1}{s}-\frac{\beta}{n}}\left(\frac{1}{|2^{k+1}Q|^{1-\frac{s\beta}{n}}}\int_{2^{k+1}Q}|f(y)|^s dy\right)^{1/s}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\sum_{k=1}^{\infty}C_k\sup_{y\in 2^{k+1}Q}|b_1(y)-(b_1)_{2^{k+1}Q}||2^{k+1}Q|^{-\frac{\beta}{n}}M_{\beta,s}(f)(\tilde{x})\\
&+ C\sum_{k=1}^{\infty}C_k|(b_1)_{2^{k+1}Q}-(b_1)_{2Q}||2^{k+1}Q|^{-\frac{\beta}{n}}M_{\beta,s}(f)(\tilde{x})\\
&\leq C\sum_{k=1}^{\infty}C_k||b_1||_{Lip_\beta(w)}w(2^{k+1}Q)^{1+\frac{\beta}{n}}|2^{k+1}Q|^{-1}|2^{k+1}Q|^{-\frac{\beta}{n}}M_{\beta,s}(f)(\tilde{x})\\
&+ C\sum_{k=1}^{\infty}C_k k w(\tilde{x})w(2^{k+1}Q)^{\frac{\beta}{n}}||b_1||_{Lip_\beta(w)}|2^{k+1}Q|^{-\frac{\beta}{n}}M_{\beta,s}(f)(\tilde{x})\\
&\leq C||b_1||_{Lip_\beta(w)}\sum_{k=1}^{\infty}kC_k\left(\frac{w(2^{k+1}Q)}{|2^{k+1}Q|}\right)^{1+\frac{\beta}{n}}M_{\beta,s}(f)(\tilde{x})\\
&\quad +C||b_1||_{Lip_\beta(w)}w(\tilde{x})\sum_{k=1}^{\infty}kC_k\left(\frac{w(2^{k+1}Q)}{|2^{k+1}Q|}\right)^{\frac{\beta}{n}}M_{\beta,s}(f)(\tilde{x})\\
&\leq C||b_1||_{Lip_\beta(w)}w(\tilde{x})^{1+\frac{\beta}{n}}M_{\beta,s}(f)(\tilde{x}),
\end{aligned}
$$

thus

$$\frac{1}{|Q|}\int_Q C(x)dx\leq C||b_1||_{Lip_\beta(w)}w(\tilde{x})^{1+\frac{\beta}{n}}M_{\beta,s}(f)(\tilde{x}).$$

Now, we consider the **Case** $m\geq 2$. We have, for $b=(b_1,\cdots,b_m)$,

$$
\begin{aligned}
T_{\vec{b}}(f)(x) &= \int_{\mathbb{R}^n}\prod_{j=1}^m(b_j(x)-b_j(y))K(x,y)f(y)dy\\
&= \int_{\mathbb{R}^n}\prod_{j=1}^m[(b_j(x)-(b_j)_{2Q})-(b_j(y)-(b_j)_{2Q})]K(x,y)f(y)dy\\
&= \sum_{j=0}^m\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma\int_{R^n}(b(y)-(b)_{2Q})_{\sigma^c}K(x,y)f(y)dy\\
&= \prod_{j=1}^m(b_j(x)-(b_j)_{2Q})\int_{\mathbb{R}^n}K(x,y)f(y)dy\\
&+ (-1)^m\int_{\mathbb{R}^n}\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})K(x,y)f(y)dy\\
&+ \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b_j(x)-(b_j)_{2Q})_\sigma\int_{\mathbb{R}^n}(b_j(y)-(b_j)_{2Q})_{\sigma^c}K(x,y)f(y)dy\\
&= \prod_{j=1}^m(b_j(x)-(b_j)_{2Q})T(f)(x)+(-1)^mT(\prod_{j=1}^m(b_j-(b_j)_{2Q})f)(x)\\
&+ \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}((b_j(x)-(b_j)_{2Q})_\sigma)T(b_j-(b_j)_{2Q})_{\sigma^c}(f)(x),
\end{aligned}
$$

thus, recall that $C_0=T(\prod_{j=1}^m(b_j-(b_j)_{2Q})f_2)(x_0)$,

$$\begin{aligned}
&|T_{\vec{b}}(f)(x) - T(\prod_{j=1}^{m}(b_j - (b_j)_{2Q})f_2)(x_0)| \\
&\le \quad |\prod_{j=1}^{m}(b_j(x) - (b_j)_{2Q})T(f)(x)| + |T(\prod_{j=1}^{m}(b_j - (b_j)_{2Q})f_1)(x)| \\
&+ \quad |\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}((b_j(x) - (b_j)_{2Q})_\sigma)T(b_j - (b_j)_{2Q})_{\sigma^c}(f)(x)| \\
&+ \quad |T(\prod_{j=1}^{m}(b_j - (b_j)_{2Q})f_2)(x) - T(\prod_{j=1}^{m}(b_j - (b_j)_{2Q})f_2)(x_0)| \\
&= \quad I_1(x) + I_2(x) + I_3(x) + I_4(x).
\end{aligned}$$

where

$$\begin{aligned}
I_1(x) &= |\prod_{j=1}^{m}(b_j(x) - (b_j)_{2Q})T(f)(x)|, \\
I_2(x) &= |T(\prod_{j=1}^{m}(b_j - (b_j)_{2Q})f_1)(x)|, \\
I_3(x) &= |\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}((b_j(x) - (b_j)_{2Q})_\sigma)T(b_j - (b_j)_{2Q})_{\sigma^c}(f)(x)|, \\
I_4(x) &= |T(\prod_{j=1}^{m}(b_j - (b_j)_{2Q})f_2)(x) - T(\prod_{j=1}^{m}(b_j - (b_j)_{2Q})f_2)(x_0)|.
\end{aligned}$$

For $I_1(x)$, by Hölder's inequality with exponent $\frac{1}{r_1} + \cdots + \frac{1}{r_m} + \frac{1}{s} = 1$ and Lemma 3.2, we get

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q I_1(x)dx \le C\frac{1}{|Q|}\int_Q |\prod_{j=1}^{m}(b_j(x) - (b_j)_{2Q})||T(f)(x)|dx \\
&\le \quad C\frac{1}{|Q|}\prod_{j=1}^{m}\left(\int_Q ||b_j(x) - (b_j)_{2Q}|^{r_j}dx\right)^{\frac{1}{r_j}}\left(\int_Q |T(f)(x)|^s dx\right)^{\frac{1}{s}} \\
&\le \quad C\frac{1}{|Q|}\prod_{j=1}^{m}\left(\sup_{x\in Q}|b_j(x) - (b_j)_{2Q}||Q|^{\frac{1}{r_j}}\right)\left(\int_Q |T(f)(x)|^s dx\right)^{\frac{1}{s}} \\
&\le \quad C\frac{1}{|Q|}\prod_{j=1}^{m}(||b_j||_{Lip_\beta(w)}w(Q)^{1+\frac{\beta}{n}}|Q|^{-1})|Q|^{(1-\frac{1}{s})+(\frac{1}{s}-\frac{m\beta}{n})}\left(\frac{1}{|Q|^{1-\frac{rm\beta}{n}}}\int_Q |T(f)(x)|^s dx\right)^{\frac{1}{s}} \\
&\le \quad C||\vec{b}||_{Lip_\beta(w)}w(Q)^{m+\frac{m\beta}{n}}|Q|^{-m-\frac{m\beta}{n}}M_{m\beta,s}(T(f))(\tilde{x}) \\
&\le \quad C||\vec{b}||_{Lip_\beta(w)}\left(\frac{w(Q)}{|Q|}\right)^{m+\frac{m\beta}{n}}M_{m\beta,s}(T(f))(\tilde{x}) \\
&\le \quad C||\vec{b}||_{Lip_\beta(w)}w(\tilde{x})^{m+\frac{m\beta}{n}}M_{m\beta,s}(T(f))(\tilde{x}).
\end{aligned}$$

For $I_2(x)$, similar to $B(x)$, using the boundness of T and Lemma 3.2, we get, for $1 < t < s$,

$$\frac{1}{|Q|}\int_Q I_2(x)dx \leq C\frac{1}{|Q|}\left(\int_{\mathbb{R}^n}|T(\prod_{j=1}^m(b_j-(b_j)_{2Q})f_1)(x)|^t dx\right)^{\frac{1}{t}}|Q|^{\frac{1}{t'}}$$

$$\leq C\frac{1}{|Q|}\left(\int_{\mathbb{R}^n}|\prod_{j=1}^m(b_j(x)-(b_j)_{2Q})f_1(x)|^r dx\right)^{\frac{1}{t}}|Q|^{\frac{1}{t'}}$$

$$\leq C\frac{1}{|Q|}\left(\int_Q|\prod_{j=1}^m(b_j(x)-(b_j)_{2Q})|^s|f(x)|^s dx\right)^{\frac{1}{s}}|Q|^{\frac{1}{s'}}$$

$$\leq C\frac{1}{|Q|}(\prod_{j=1}^m(b_j(x)-(b_j)_{2Q}))(\int_{2Q}|f(x)|^s dx)^{\frac{1}{s}}|Q|^{\frac{1}{s'}}$$

$$\leq C\frac{1}{|Q|}(\prod_{j=1}^m||b_j||_{Lip_\beta(w)}w(2Q)^{1+\frac{\beta}{n}}|2Q|^{-1})|2Q|^{\frac{1}{s'}+\frac{1}{s}-\frac{m\beta}{n}}\left(\frac{1}{|2Q|^{1-\frac{sm\beta}{n}}}\int_{2Q}|f(x)|^s dx\right)^{\frac{1}{s}}$$

$$\leq C||\vec{b}||_{Lip_\beta(w)}\left(\frac{w(2Q)}{|2Q|}\right)^{m+\frac{m\beta}{n}}M_{m\beta,s}(f)(\tilde{x})$$

$$\leq C||\vec{b}||_{Lip_\beta(w)}w(\tilde{x})^{m+\frac{m\beta}{n}}M_{m\beta,s}(f)(\tilde{x}).$$

For $I_3(x)$, by Hölder's inequality and Lemma 3.2, we get

$$\frac{1}{|Q|}\int_Q I_3(x)dx \leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|2Q|}\int_{2Q}|(b_j(x)-(b_j)_{2Q})_\sigma||T((b_j-(b_j)_{2Q})_{\sigma^c}f)(x)|dx$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|2Q|}\left(\int_{2Q}|(b_j(x)-(b_j)_{2Q})_\sigma|^{s'}dx\right)^{\frac{1}{s'}}$$

$$\times \left(\int_{2Q}|T(b_j-(b_j)_{2Q})_{\sigma^c}(f)(x)|^s dx\right)^{\frac{1}{s}}$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|2Q|}\sup_{x\in 2Q}|(b_j(x)-(b_j)_{2Q})_\sigma||2Q|^{\frac{1}{r'}}$$

$$\times \left(\int_{2Q}|(b_j(x)-(b_j)_{2Q})_{\sigma^c}|^s|T(f)(x)|^s dx\right)^{\frac{1}{s}}$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|2Q|}\sup_{x\in 2Q}|(b_j(x)-(b_j)_{2Q})_\sigma||2Q|^{\frac{1}{r'}}$$

$$\times \sup_{x\in 2Q}|(b_j(x)-(b_j)_{2Q})_{\sigma^c}|\left(\int_{2Q}|T(f)(x)|^s dx\right)^{\frac{1}{s}}$$

$$\leq C\frac{1}{|2Q|}||\vec{b_\sigma}||_{Lip_\beta(w)}w(2Q)^{j+\frac{j\beta}{n}}|2Q|^{-j}||\vec{b_{\sigma^c}}||_{Lip_\beta(w)}w(2Q)^{(m-j)+\frac{(m-j)\beta}{n}}$$

$$\times |2Q|^{\frac{1}{s'}+\frac{1}{s}-\frac{m\beta}{n}-(m-j)}\left(\frac{1}{|2Q|^{1-\frac{sm\beta}{n}}}\int_{2Q}|T(f)(x)|^s dx\right)^{\frac{1}{s}}$$

$$
\begin{aligned}
&\le C||\vec{b}||_{Lip_\beta(w)}\left(\frac{w(2Q)}{|2Q|}\right)^{m+\frac{m\beta}{n}} M_{m\beta,s}(T(f))(\tilde{x}) \\
&\le C||\vec{b}||_{Lip_\beta(w)} w(\tilde{x})^{m+\frac{m\beta}{n}} M_{m\beta,s}(T(f))(\tilde{x}).
\end{aligned}
$$

For $I_4(x)$, similar to the proof of $C(x)$ in the **Case** $m = 1$, for $1 < p < \infty, 1 < t < s$ with $1/p + 1/q + 1/t = 1$, we have

$$
\begin{aligned}
&|T(\prod_{j=1}^{m}((b_j(y)-(b_j)_{2Q})f_2)(x) - T(\prod_{j=1}^{m}((b_j-(b_j)_{2Q})f_2)(x_0)| \\
&\le \int_{(2Q)^c} |\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})||f(y)||(K(x,y)-K(x_0,y))|dy \\
&\le C\sum_{k=1}^{\infty}\left(\int_{2^k|x-x_0|\le|y-x_0|<2^{k+1}|x-x_0|}|K(x,y)-K(x_0,y)|^q dy\right)^{1/q} \\
&\times \left(\int_{|y-x_0|<2^{k+1}|x-x_0|}|\prod_{j=1}^{m}b_1(y)-(b_1)_{2Q}|^p dy\right)^{1/p}\left(\int_{|y-x_0|<2^{k+1}|x-x_0|}|f(y)|^t dy\right)^{1/t} \\
&\le C\sum_{k=1}^{\infty}C_k\frac{1}{(2^k d)^{\frac{n}{q'}}}\left(\int_{|y-x_0|<2^{k+1}|x-x_0|}\prod_{j=1}^{m}|b_j(y)-(b_j)_{2Q}|^p dy\right)^{1/p} \\
&\times \left(\int_{|y-x_0|<2^{k+1}|x-x_0|}|f(y)|^t dy\right)^{1/t} \\
&\le C\sum_{k=1}^{\infty}C_k\frac{1}{(2^k d)^{\frac{n}{q'}}}\left(\int_{2^{k+1}Q}\prod_{j=1}^{m}|b_j(y)-(b_j)_{2^{k+1}Q}|^p dy\right)^{1/p}\left(\int_{2^{k+1}Q}|f(y)|^t dy\right)^{1/t} \\
&+ C\sum_{k=1}^{\infty}C_k\frac{1}{(2^k d)^{\frac{n}{q'}}}\left(\int_{2^{k+1}Q}\prod_{j=1}^{m}|(b_j)_{2^{k+1}Q}-(b_j)_{2Q}|^p dy\right)^{1/p}\left(\int_{2^{k+1}Q}|f(y)|^t dy\right)^{1/t} \\
&\le C\sum_{k=1}^{\infty}C_k\frac{1}{|2^kQ|^{\frac{1}{q'}}}\prod_{j=1}^{m}\sup_{x\in 2^{k+1}Q}|b_j(y)-(b_j)_{2^{k+1}Q}||2^{k+1}Q|^{\frac{1}{p}}|2^{k+1}Q|^{\frac{1}{s}-\frac{m\beta}{n}} \\
&\times \left(\frac{1}{|2^{k+1}Q|^{1-\frac{sm\beta}{n}}}\int_{2^{k+1}Q}|f(y)|^s dy\right)^{1/s} + C\sum_{k=1}^{\infty}C_k\frac{1}{|2^kQ|^{\frac{1}{q'}}}\prod_{j=1}^{m}|(b_j)_{2^{k+1}Q}-(b_j)_{2Q}| \\
&\times |2^{k+1}Q|^{\frac{1}{p}}|2^{k+1}Q|^{\frac{1}{s}-\frac{m\beta}{n}}\left(\frac{1}{|2^{k+1}Q|^{1-\frac{sm\beta}{n}}}\int_{2^{k+1}Q}|f(y)|^s dy\right)^{1/s} \\
&\le C\sum_{k=1}^{\infty}C_k||\vec{b}||_{Lip_\beta(w)}w(2^{k+1}Q)^{m+\frac{m\beta}{n}}|2^{k+1}Q|^{-m}|2^{k+1}Q|^{-\frac{m\beta}{n}}M_{m\beta,s}(f)(\tilde{x}) \\
&+ C\sum_{k=1}^{\infty}C_k k w(\tilde{x})^m w(2^{k+1}Q)^{\frac{m\beta}{n}}||\vec{b}||_{Lip_\beta(w)}|2^{k+1}Q|^{-\frac{m\beta}{n}}M_{m\beta,s}(f)(\tilde{x})
\end{aligned}
$$

$$\begin{aligned}
&\le C||\vec{b}||_{Lip_\beta(w)}\sum_{k=1}^{\infty}k^mC_k\left(\frac{w(2^{k+1}Q)}{|2^{k+1}Q|}\right)^{m+\frac{m\beta}{n}}M_{m\beta,s}(f)(\tilde{x})\\
&+ C||\vec{b}||_{Lip_\beta(w)}\sum_{k=1}^{\infty}k^mC_kw(\tilde{x})^m\left(\frac{w(2^{k+1}Q)}{|2^{k+1}Q|}\right)^{\frac{m\beta}{n}}M_{m\beta,s}(f)(\tilde{x})\\
&\le C||\vec{b}||_{Lip_\beta(w)}w(\tilde{x})^{m+\frac{m\beta}{n}}M_{m\beta,s}(f)(\tilde{x}),
\end{aligned}$$

thus, we get

$$\frac{1}{|Q|}\int_Q I_4(x)dx \le C||\vec{b}||_{Lip_\beta(w)}w(\tilde{x})^{m+\frac{m\beta}{n}}M_{m\beta,s}(f)(\tilde{x}).$$

Combining all the estimates above, we get

$$\frac{1}{|Q|}\int_Q |T_{\vec{b}}(f)(x)-C_0|dx \le C||\vec{b}||_{Lip_\beta(w)}w(\tilde{x})^{m+\frac{m\beta}{n}}(M_{m\beta,s}(f)(\tilde{x})+M_{m\beta,s}(T(f))(\tilde{x}))$$

and

$$T_{\vec{b}}(f)^{\#}(\tilde{x}) \le C||\vec{b}||_{Lip_\beta(w)}w(\tilde{x})^{m+\frac{m\beta}{n}}(M_{m\beta,s}(f)(\tilde{x})+M_{m\beta,s}(T(f))(\tilde{x})).$$

Now, choose $q' < s < r$, by Lemma 3.5, we have

$$\begin{aligned}
||T_{\vec{b}}(f)||_{L^q(w^{1-m+(q-1)\frac{m\beta}{n}})} &\le C||M(T_{\vec{b}}(f))||_{L^q(w^{1-m+(q-1)\frac{m\beta}{n}})}\\
&\le C||(T_{\vec{b}}(f))^{\#}||_{L^q(w^{1-m+(q-1)\frac{m\beta}{n}})}\\
&\le C||\vec{b}||_{Lip_\beta(w)}(||w^{m+m\beta/n}M_{m\beta,s}(f)||_{L^q(w^{1-m+(q-1)\frac{m\beta}{n}})}\\
&+ ||w^{m+m\beta/n}M_{m\beta,s}(T(f))||_{L^q(w^{1-m+(q-1)\frac{m\beta}{n}})})\\
&\le C||\vec{b}||_{Lip_\beta(w)}(||M_{m\beta,s}(f)||_{L^q(w^{\frac{q}{p}})}+||M_{m\beta,s}(T(f))||_{L^q(w^{\frac{q}{p}})})\\
&\le C||\vec{b}||_{Lip_\beta(w)}(||f||_{L^p(w)}+||T(f)||_{L^p(w)})\\
&\le C||\vec{b}||_{Lip_\beta(w)}||f||_{L^p(w)}.
\end{aligned}$$

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. Similar to Theorem 2.1, for any $q' < s < \infty$ and cube Q, there exists some constant C_0 such that for $f \in L^p(w)$ and $\tilde{x} \in Q$,

$$|Q|^{-1-\frac{m\beta}{n}}\int_Q |T_{\vec{b}}(f)(x)-C_0|dx \le C||\vec{b}||_{Lip_\beta(w)}w(\tilde{x})^{m+\frac{m\beta}{n}}(M_s(f)(\tilde{x})+M_s(T(f))(\tilde{x})).$$

Further, we have

$$\sup_{Q\ni\tilde{x}}\inf_{c\in\mathcal{C}}|Q|^{-1-\frac{m\beta}{n}}\int_Q |T_{\vec{b}}(f)(x)-c|dx \le C||\vec{b}||_{Lip_\beta(w)}w(\tilde{x})^{m+\frac{m\beta}{n}}(M_s(f)(\tilde{x})+M_s(T(f))(\tilde{x})).$$

Choose $q' < s < p$ and by Lemma 3.4, we obtain

$$\begin{aligned}
&||T_{\vec{b}}(f)||_{\dot{F}_p^{m\beta,\infty}(w^{1-m-\frac{m\beta}{n}})} \approx \left|\left|\sup_{\tilde{x}\in Q}\inf_{c\in\mathcal{C}}|Q|^{-1-\frac{m\beta}{n}}\int_Q |T_{\vec{b}}(f)(x)-c|dx\right|\right|_{L^p(w^{1-m-\frac{m\beta}{n}})}\\
&\le C||\vec{b}||_{Lip_\beta(w)}(||w^{m+m\beta/n}M_s(f)||_{L^p(w^{1-m-\frac{m\beta}{n}})}+||w^{m+\frac{m\beta}{n}}M_s(T(f))||_{L^p(w^{1-m-\frac{m\beta}{n}})})\\
&\le C||\vec{b}||_{Lip_\beta(w)}(||M_s(f)||_{L^p(w)}+||M_s(T(f))||_{L^p(w)})\\
&\le C||\vec{b}||_{Lip_\beta(w)}(||f||_{L^p(w)}+||T(f)||_{L^p(w)})\\
&\le C||\vec{b}||_{Lip_\beta(w)}||f||_{L^p(w)}.
\end{aligned}$$

This completes the proof of Theorem 2.2. □

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