DOI: 10.2478/awutm-2013-0004

VERSITA

Analele Universităţii de Vest,
Timişoara
Seria Matematică – Informatică
LI, 1, (2013), 47– 62

Continuity for Multilinear Commutator of Singular Integral Operator with General Kernel on Besov Spaces

Sheng Guo, Chuangxia Huang, and Lanzhe Liu

Abstract. In this paper, we prove the the continuity for the multilinear commutator associated to the singular integral operator with general kernel on the Besov spaces.

AMS Subject Classification (2000). 42B20, 42B25.
Keywords. Multilinear commutator; Singular integral operator; Besov space.

1 Introduction

With the development of the singular integral operators, their commutators and multilinear operators have been well studied. In [3][6][9-11], we know that the commutators and multilinear operators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$ are proved by others. In [2][6][8], the boundedness for the commutators and multilinear operators generated by the singular integral operators and Lipschitz functions on $L^p(R^n)(1 < p < \infty)$ and Triebel-Lizorkin spaces are obtained. In this paper, we will prove the continuity properties for the multilinear commutators related to the singular integral operator with general kernel on the Besov space.

2 Notations and Results

Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$f^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)dx$. It is well-known that (see [4][12])

$$f^{\#}(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

For $\beta \geq 0$, the Besov space $\dot{\Lambda}_\beta(R^n)$ is the space of functions f such that

$$||f||_{\dot{\Lambda}_\beta} = \sup_{\substack{x, h \in R^n \\ h \neq 0}} \left| \Delta_h^{[\beta]+1} f(x) \right| / |h|^\beta < \infty,$$

where Δ_h^k denotes the k-th difference operator(see [8]).
For $b_j \in \dot{\Lambda}_\beta(R^n)$ $(j = 1, \cdots, m)$, set

$$||\vec{b}||_{\dot{\Lambda}_\beta} = \prod_{j=1}^m ||b_j||_{\dot{\Lambda}_\beta}.$$

Given some functions b_j $(j = 1, \cdots, m)$ and a positive integer m and $1 \leq j \leq m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{\dot{\Lambda}_\beta} = ||b_{\sigma(1)}||_{\dot{\Lambda}_\beta} \cdots ||b_{\sigma(j)}||_{\dot{\Lambda}_\beta}$.
To state our results, we first give some definitions(see [1]).
Definition 1. *Let $T : S \to S'$ be a linear operator such that T is bounded on $L^2(R^n)$ and there exists a locally integrable function $K(x, y)$ on $R^n \times R^n \setminus \{(x, y) \in R^n \times R^n : x = y\}$ such that*

$$T(f)(x) = \int_{R^n} K(x, y) f(y) dy$$

for every bounded and compactly supported function f, where K satisfies: there is a sequence of positive constant numbers $\{C_k\}$ such that for any $k \geq 1$,

$$\int_{2|y-z|<|x-y|} (|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)|) dx \leq C,$$

and

$$l\left(\int_{2^k|z-y|\le|x-y|<2^{k+1}|z-y|}(|K(x,y)-K(x,z)|+|K(y,x)-K(z,x)|)^q dy\right)^{1/q}\le C_k(2^k|z-y|)^{-n/q'},$$

where $1<q'<2$ *and* $1/q+1/q'=1$. *Suppose* b_j $(j=1,\cdots,m)$ *are the fixed locally integrable functions on* R^n. *The multilinear commutator of the singular integral operator is defined by*

$$T_{\vec{b}}(f)(x)=\int_{R^n}\prod_{j=1}^m(b_j(x)-b_j(y))K(x,y)f(y)dy.$$

Note that the classical Calderón-Zygmund singular integral operator satisfies **Definition 1** with $C_k=2^{-k\delta}$(see [1][7][12]). Also note that when $m=1$, $T_{\vec{b}}$ is just the commutator what we mentioned above. It is well known that multilinear operator are of great interest in harmonic analysis and have been widely studied by many authors (see [9-11]). In [10], Pérez and Trujillo-Gonzalez prove a sharp estimate for the multilinear commutator.

Definition 2. *Let* $0<p,q\le\infty$, $\alpha\in R$. *For* $k\in Z$, *set* $B_k=\{x\in R^n: |x|\le 2^k\}$ *and* $C_k=B_k\setminus B_{k-1}$. *Denote by* χ_k *the characteristic function of* C_k *and* χ_0 *the characteristic function of* B_0.

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,\ p}(R^n)=\{f\in L^q_{loc}(R^n\setminus\{0\}):||f||_{\dot{K}_q^{\alpha,\ p}}<\infty\},$$

where

$$||f||_{\dot{K}_q^{\alpha,\ p}}=\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}||f\chi_k||^p_{L^q}\right]^{1/p};$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,\ p}(R^n)=\{f\in L^q_{loc}(R^n):||f||_{K_q^{\alpha,\ p}}<\infty\},$$

where

$$||f||_{K_q^{\alpha,\ p}}=\left[\sum_{k=1}^{\infty}2^{k\alpha p}||f\chi_k||^p_{L^q}+||f\chi_{B_0}||^p_{L^q}\right]^{1/p};$$

And the usual modification is made when $p=q=\infty$.

Definition 3. *Let* $1\le q<\infty$, $\alpha\in R$. *The central Campanato space is defined by*

$$CL_{\alpha,\ q}(R^n)=\{f\in L^q_{loc}(R^n):||f||_{CL_{\alpha,\ q}}<\infty\},$$

where

$$||f||_{CL_{\alpha,\ q}} = \sup_{r>0} |B(0,r)|^{-\alpha} \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x) - f_{B(0,r)}|^q dx \right)^{1/q}.$$

Now we state our theorems as following.

Theorem 1. *Let $0 < \beta < \frac{1}{m}$ and $b_j \in \dot{\Lambda}_\beta(R^n)$ for $j = 1, \cdots, m$. Then $T_{\vec{b}}$ is bounded from $L^p(R^n)$ to $\dot{\Lambda}_{\frac{m\beta}{n}-\frac{1}{p}}(R^n)$ for any p with $\max(q', n/m\beta) \le p < \infty$.*

Theorem 2. *Let $0 < \beta < \frac{1}{m}$, $1 < q_1 < \frac{n}{m\beta}$, $\frac{1}{q_2} = \frac{1}{q_1} - \frac{m\beta}{n}$, $-\frac{n}{q_2} - 1 < \alpha \le -\frac{n}{q_2}$ and $b_j \in \dot{\Lambda}_\beta(R^n)$ for $j = 1, \cdots, m$. Then $T_{\vec{b}}$ is bounded from $\dot{K}^{\alpha,\infty}_{q_1}(R^n)$ to $CL_{-\frac{\alpha}{n}-\frac{1}{q_2}, q_2}(R^n)$.*

Remark. Theorem 2 also hold for the nonhomogeneous Herz type Hardy space.

3 Proofs of Theorems

To prove our theorems, we need the following lemmas.

Lemma 1.(see [8]) *For $0 < \beta < 1, 1 \le p \le \infty$, we have*

$$||b||_{\dot{\Lambda}_\beta} \approx \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - b_Q| dx \approx \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |b(x) - b_Q|^p dx \right)^{1/p}$$

$$\approx \sup_Q \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - c| dx \approx \sup_Q \inf_c \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |b(x) - c|^p dx \right)^{1/p}.$$

Lemma 2.(see [13]) *For $\alpha < 0, 0 < q < \infty$, we have*

$$||f||_{\dot{K}^{\alpha,\ \infty}_q} \approx \sup_{\mu \in Z} 2^{\mu\alpha} ||f\chi_{B_\mu}||_{L^q}.$$

Lemma 3. *Let $0 < \eta < n$, $1 < p < n/\eta$. Suppose $b \in \dot{\Lambda}_\beta(R^n)$, then*

$$|b_{2^{k+1}B} - b_B| \le C||b||_{\dot{\Lambda}_\beta} k |2^{k+1}B|^{\beta/n} \text{ for } k \ge 1\ .$$

Proof.

$$\begin{aligned}
|b_{2^{k+1}B} - b_B| &\le \sum_{j=0}^{k} |b_{2^{j+1}B} - b_{2^jB}| \\
&\le \sum_{j=0}^{k} \frac{1}{|2^jB|}\int_{2^jB} |b(y) - b_{2^{j+1}B}|dy \\
&\le C\sum_{j=0}^{k} \left(\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^p dy\right)^{1/p} \\
&\le C||b||_{\dot\Lambda_\beta} \sum_{j=0}^{k} |2^{j+1}B|^{\beta/n} \\
&\le C||b||_{\dot\Lambda_\beta}(k+1)|2^{k+1}B|^{\beta/n} \\
&\le C||b||_{\dot\Lambda_\beta} k|2^{k+1}B|^{\beta/n}.
\end{aligned}$$

Lemma 4.(see [5][7]) *Let* $0 \le \beta < 1$, $1 < r < n/\beta$, $1/r - 1/s = \beta/n$ *and* $b_j \in \dot\Lambda_\beta(R^n)$ *for* $j = 1, \cdots, m$. *Then* $T_{\vec b}$ *is bounded from* $L^r(R^n)$ *to* $L^s(R^n)$.

Proof of Theorem 1. It is enough to prove that there exists a constant C_0 such that

$$\frac{1}{|Q|^{1+m\beta/n-1/p}}\int_Q |T_{\vec b}(f)(x) - C_0|dx \le C||f||_{L^p}.$$

Fix a ball Q, $Q = Q(x_0, r)$, we decompose f into $f = f_1 + f_2$ with $f_1 = f\chi_{2Q}$, $f_2 = f\chi_{(R^n\setminus 2Q)}$.

Following [5], we will consider the cases $m = 1$ and $m > 1$, respectively.

We first consider the **Case** $m = 1$. For $C_0 = T(((b_1)_{2Q} - b_1)f_2)(x_0)$, we have

$$T_{b_1}(f)(x) = (b_1(x) - (b_1)_{2Q})T(f)(x) - T((b_1 - (b_1)_{2Q})f)(x).$$

Then

$$\begin{aligned}
&|T_{b_1}(f)(x) - C_0| \\
\le\ & |(b_1(x) - (b_1)_{2Q})T(f)(x)| \\
&+|T(((b_1)_{2Q} - b_1)f_1)(x)| \\
&+|T(((b_1)_{2Q} - b_1)f_2)(x) - T(((b_1)_{2Q} - b_1)f_2)(x_0)| \\
=\ & A_1(x) + A_2(x) + A_3(x).
\end{aligned}$$

For $A_1(x)$, by the boundedness of T and Hölder's inequality with $\frac{1}{p'}+\frac{1}{p} = 1$,we have

$$\begin{aligned}
&\frac{1}{|2Q|^{1+\frac{\beta}{n}-\frac{1}{p}}}\int_{2Q}A_1(x)dx\\
\leq\ & C\frac{1}{|2Q|^{1+\frac{\beta}{n}-\frac{1}{p}}}\left(\int_{2Q}|(b_1(x)-(b_1)_{2Q})|^{p'}dx\right)^{\frac{1}{p'}}\left(\int_{2Q}|T(f)(x)|^{p}dx\right)^{\frac{1}{p}}\\
\leq\ & C\frac{|2Q|^{\frac{\beta}{n}+\frac{1}{p'}}}{|2Q|^{1+\frac{\beta}{n}-\frac{1}{p}}}\frac{1}{|2Q|^{\frac{\beta}{n}}}\left(\frac{1}{|2Q|}\int_{2Q}|(b_1(x)-(b_1)_{2Q})|^{p'}dx\right)^{\frac{1}{p'}}\left(\int_{2Q}|f(x)|^{p}dx\right)^{\frac{1}{p}}\\
\leq\ & C||b_1||_{\dot{\Lambda}_\beta}||f||_{L^p}.
\end{aligned}$$

For $A_2(x)$, taking $1<r<p<\infty$ and $p=rt$, by Hölder's inequality, we have

$$\begin{aligned}
&\frac{1}{|2Q|^{1+\frac{\beta}{n}-\frac{1}{p}}}\int_{2Q}A_2(x)dx\\
\leq\ & C\frac{1}{|2Q|^{1+\frac{\beta}{n}-\frac{1}{p}}}\left(\int_{2Q}|T(((b_1)_{2Q}-b_1)f\chi_{2Q})(x)|^{r}dx\right)^{\frac{1}{r}}|2Q|^{\frac{1}{r'}}\\
\leq\ & C\frac{1}{|2Q|^{\frac{\beta}{n}-\frac{1}{p}}}\frac{1}{|2Q|^{\frac{1}{r}}}\left(\int_{2Q}|((b_1)_{2Q}-b_1)f(x)|^{r}dx\right)^{\frac{1}{r}}\\
\leq\ & C\frac{1}{|2Q|^{\frac{\beta}{n}-\frac{1}{p}+\frac{1}{r}}}\left(\int_{2Q}|((b_1)_{2Q}-b_1)|^{rq'}dx\right)^{\frac{1}{rq'}}\left(\int_{2Q}|f(x)|^{rq}dx\right)^{\frac{1}{rq}}\\
\leq\ & C\frac{|2Q|^{\frac{\beta}{n}+\frac{1}{rq'}}}{|2Q|^{\frac{\beta}{n}-\frac{1}{p}+\frac{1}{r}}}\frac{1}{|2Q|^{\beta}}\left(\frac{1}{|2Q|}\int_{2Q}|((b_1)_{2Q}-b_1)|^{rq'}dx\right)^{\frac{1}{rq'}}||f||_{L^p}\\
\leq\ & C||b_1||_{\dot{\Lambda}_\beta}||f||_{L^p}.
\end{aligned}$$

For $A_3(x)$, taking $1<q<\infty$ with $1/p+1/q+1/t=1$, for $x\in Q$, by Hölder's inequality and Lemma 3, we have

$$
\begin{aligned}
& |T(b_1-(b_1)_{2Q})(f_2)(x)-T(b_1-(b_1)_{2Q})(f_2)(x_0)| \\
= & \left|\int_{(2Q)^c}(b_1(y)-(b_1)_{2Q})f_2(y)(K(x,y)-K(x_0,y))dy\right| \\
\leq & \int_{(2Q)^c}|(b_1(y)-(b_1)_{2Q})||f(y)||(K(x,y)-K(x_0,y))|dy \\
\leq & C\sum_{k=1}^{\infty}\left(\int_{2^k|x-x_0|\leq|y-x_0|<2^{k+1}|x-x_0|}|K(x,y)-K(x_0,y)|^t dy\right)^{1/t} \\
& \times\left(\int_{|y-x_0|<2^{k+1}|x-x_0|}|b_1(y)-(b_1)_{2Q}|^q dy\right)^{1/q}\left(\int_{|y-x_0|<2^{k+1}|x-x_0|}|f(y)|^p dy\right)^{1/p} \\
\leq & C\sum_{k=1}^{\infty}C_k\frac{1}{(2^kd)^{\frac{n}{t'}}}\left(\int_{|y-x_0|<2^{k+1}|x-x_0|}|b_1(y)-(b_1)_{2Q}|^q dy\right)^{1/q} \\
& \times\left(\int_{|y-x_0|<2^{k+1}|x-x_0|}|f(y)|^p dy\right)^{1/p} \\
\leq & C\sum_{k=1}^{\infty}C_k\frac{1}{(2^kd)^{\frac{n}{t'}}}\left(\int_{2^{k+1}Q}|b_1(y)-(b_1)_{2Q}|^q dy\right)^{1/q}\left(\int_{2^{k+1}Q}|f(y)|^p dy\right)^{1/p} \\
\leq & C\sum_{k=1}^{\infty}C_k\frac{1}{(2^kd)^{\frac{n}{t'}}}\Bigg[\left(\int_{2^{k+1}Q}|b_1(y)-(b_1)_{2^{k+1}Q}|^q dx\right)^{\frac{1}{q}} \\
& \qquad\qquad +|(b_1)_{2^{k+1}Q}-(b_1)_{2Q}||2^{k+1}Q|^{\frac{1}{q}}\Bigg]||f||_{L^p} \\
\leq & C\sum_{k=1}^{\infty}C_k\frac{1}{|2^{k+1}Q|^{\frac{1}{t'}}}\Bigg[\frac{1}{|2^{k+1}Q|^{\frac{\beta}{n}}}\left(\frac{1}{2^{k+1}Q}\int_{2^{k+1}Q}|b_1(y)-(b_1)_{2^{k+1}Q}|^q dx\right)^{\frac{1}{q}}|2^{k+1}Q|^{\frac{\beta}{n}+\frac{1}{q}} \\
& \qquad\qquad +|(b_1)_{2^{k+1}Q}-(b_1)_{2Q}||2^{k+1}Q|^{\frac{1}{q}}\Bigg]||f||_{L^p} \\
\leq & C\sum_{k=1}^{\infty}C_k\frac{1}{|2^{k+1}Q|^{\frac{1}{t'}}}\left[|2^{k+1}Q|^{\frac{\beta}{n}+\frac{1}{q}}||b_1||_{\dot\Lambda_\beta}+Ck||b_1||_{\dot\Lambda_\beta}|2^{k+1}Q|^{\frac{\beta}{n}+\frac{1}{q}}\right]||f||_{L^p} \\
\leq & C\sum_{k=1}^{\infty}C_k|2^{k+1}Q|^{\frac{\beta}{n}-\frac{1}{p}}||b_1||_{\dot\Lambda_\beta}||f||_{L^p} \\
\leq & C|2^{k+1}Q|^{\frac{\beta}{n}-\frac{1}{p}}||b_1||_{\dot\Lambda_\beta}||f||_{L^p},
\end{aligned}
$$

thus

$$
\frac{1}{|2Q|^{1+\frac{\beta}{n}-\frac{1}{p}}}\int_{2Q}A_3(x)dx\leq C\frac{1}{|2Q|^{1+\frac{\beta}{n}-\frac{1}{p}}}|2Q|^{\frac{\beta}{n}-\frac{1}{p}}||b_1||_{\dot\Lambda_\beta}||f||_{L^p}|2Q|\leq C||b_1||_{\dot\Lambda_\beta}||f||_{L^p}.
$$

Now, we consider the **Case** $m \geq 2$. We have, for $b = (b_1, \cdots, b_m)$,

$$
\begin{aligned}
T_{\vec{b}}(f)(x) &= \int_{R^n} \prod_{j=1}^{m} (b_j(x) - b_j(y)) K(x,y) f(y) dy \\
&= \int_{R^n} \prod_{j=1}^{m} [(b_j(x) - (b_j)_{2Q}) - (b_j(y) - (b_j)_{2Q})] K(x,y) f(y) dy \\
&= \sum_{j=0}^{m} \sum_{\sigma \in C_j^m} (-1)^{m-j} (b(x) - (b)_{2Q})_\sigma \int_{R^n} (b(y) - (b)_{2Q})_{\sigma^c} K(x,y) f(y) dy \\
&= \prod_{j=1}^{m} (b_j(x) - (b_j)_{2Q}) \int_{R^n} K(x,y) f(y) dy \\
&\quad + (-1)^m \int_{R^n} \prod_{j=1}^{m} (b_j(y) - (b_j)_{2Q}) K(x,y) f(y) dy \\
&\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} (b_j(x) - (b_j)_{2Q})_\sigma \int_{R^n} (b_j(y) - (b_j)_{2Q})_{\sigma^c} K(x,y) f(y) dy \\
&= \prod_{j=1}^{m} (b_j(x) - (b_j)_{2Q}) T(f)(x) + (-1)^m T(\prod_{j=1}^{m} (b_j - (b_j)_{2Q}) f)(x) \\
&\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} ((b_j(x) - (b_j)_{2Q})_\sigma) T(b_j - (b_j)_{2Q})_{\sigma^c}(f)(x),
\end{aligned}
$$

thus, recall that $C_0 = T(\prod_{j=1}^{m} (b_j - (b_j)_{2Q}) f_2)(x_0)$,

$$
\begin{aligned}
& |T_{\vec{b}}(f)(x) - T(\prod_{j=1}^{m} (b_j - (b_j)_{2Q}) f_2)(x_0)| \\
\leq \; & |\prod_{j=1}^{m} (b_j(x) - (b_j)_{2Q}) T(f)(x)| \\
& + |T(\prod_{j=1}^{m} (b_j - (b_j)_{2Q}) f_1)(x)| \\
& + |\sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} ((b_j(x) - (b_j)_{2Q})_\sigma) T(b_j - (b_j)_{2Q})_{\sigma^c}(f)(x)| \\
& + |T(\prod_{j=1}^{m} (b_j - (b_j)_{2Q}) f_2)(x) - T(\prod_{j=1}^{m} (b_j - (b_j)_{2Q}) f_2)(x_0)| \\
= \; & I_1(x) + I_2(x) + I_3(x) + I_4(x).
\end{aligned}
$$

For $I_1(x)$, let $\frac{1}{q_1}+\frac{1}{q_2}+\cdots+\frac{1}{q_m}+\frac{1}{p}=1$, using Hölder's inequality and Lemma 1, we have

$$\begin{aligned}
&\frac{1}{|2Q|^{1+\frac{m\beta}{n}-\frac{1}{p}}}\int_{2Q}|\prod_{j=1}^{m}(b_j(x)-(b_j)_{2Q})T(f)(x)|dx\\
\leq\ & C\frac{1}{|2Q|^{1+\frac{m\beta}{n}-\frac{1}{p}}}\prod_{j=1}^{m}\left(\int_{2Q}|(b_j(x)-(b_j)_{2Q})|^{q_j}dx\right)^{\frac{1}{q_j}}\left(\int_{2Q}|T(f)(x)|^{p}dx\right)^{\frac{1}{p}}\\
\leq\ & C\frac{|2Q|^{m\beta+\frac{1}{q_1}+\frac{1}{q_2}+\cdots+\frac{1}{q_m}}}{|2Q|^{1+\frac{m\beta}{n}-\frac{1}{p}}}\prod_{j=1}^{m}\frac{1}{|2Q|^{\beta}}\left(\frac{1}{|2Q|}\int_{2Q}|(b_j(x)-(b_j)_{2Q})|^{q_j}dx\right)^{\frac{1}{q_j}}\\
&\qquad\times\left(\int_{2Q}|f(x)|^{p}dx\right)^{\frac{1}{p}}\\
\leq\ & C\prod_{j=1}^{m}||b_j||_{\dot{\Lambda}_\beta}||f||_{L^p}\\
\leq\ & C||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{L^p}.
\end{aligned}$$

For $I_2(x)$, taking $1<r<p<\infty$, $p=rt$ and $\frac{1}{t_1}+\frac{1}{t_2}+\cdots+\frac{1}{t_m}+\frac{1}{t}=1$, by Hölder's inequality, we have

$$\begin{aligned}
&\frac{1}{|2Q|^{1+\frac{m\beta}{n}-\frac{1}{p}}}\int_{2Q}|T(\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})f_1)(x)|dx\\
\leq\ & \frac{1}{|2Q|^{1+\frac{m\beta}{n}-\frac{1}{p}}}\left(\int_{R^n}|T(\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})f\chi_{2Q})(x)|^{r}dx\right)^{\frac{1}{r}}|2Q|^{\frac{1}{r'}}\\
\leq\ & \frac{1}{|2Q|^{\frac{m\beta}{n}-\frac{1}{p}}}\frac{1}{|2Q|^{\frac{1}{r}}}\left(\int_{2Q}|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})f(x)|^{r}dx\right)^{\frac{1}{r}}\\
\leq\ & \frac{1}{|2Q|^{\frac{m\beta}{n}-\frac{1}{p}+\frac{1}{r}}}\prod_{j=1}^{m}\left(\int_{2Q}|(b_j-(b_j)_{2Q})|^{rt_j}dx\right)^{\frac{1}{rt_j}}\left(\int_{2Q}|f(x)|^{rt}dx\right)^{\frac{1}{rt}}\\
\leq\ & \frac{|2Q|^{\frac{m\beta}{n}+\frac{1}{rt_1}+\frac{1}{rt_2}+\cdots+\frac{1}{rt_m}}}{|2Q|^{\frac{m\beta}{n}-\frac{1}{p}+\frac{1}{r}}}\prod_{j=1}^{m}\frac{1}{|2Q|^{\beta}}\left(\frac{1}{|2Q|}\int_{2Q}|(b_j-(b_j)_{2Q})|^{rt_j}dx\right)^{\frac{1}{rt_j}}||f||_{L^p}\\
\leq\ & C\prod_{j=1}^{m}||b_j||_{\dot{\Lambda}_\beta}||f||_{L^p}\\
\leq\ & C||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{L^p}.
\end{aligned}$$

For $I_3(x)$, taking $1<r<p<\infty$, $p=rt$, and denote $1=\sum\frac{1}{t_i}$, where $\sigma(i)\in\sigma$, $\frac{1}{q}=\sum\frac{1}{s_k}$ and $\sigma(k)\in\sigma^c$, let $\frac{1}{q}+\frac{1}{t}=1$, $\lambda_1+\lambda_2=m$, by the boundedness of T and Hölder's inequality, we have

$$
\begin{aligned}
&\frac{1}{|2Q|^{1+\frac{m\beta}{n}-\frac{1}{p}}}\int_{2Q} I_3(x)dx \\
\leq\ & \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|2Q|^{1+\frac{m\beta}{n}-\frac{1}{p}}}\int_{2Q}|(b_j(x)-(b_j)_{2Q})_\sigma T((b_j-(b_j)_{2Q})_{\sigma^c}f)(x)|dx \\
\leq\ & C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|2Q|^{1+\frac{m\beta}{n}-\frac{1}{p}}}\left(\int_{2Q}|(b_j(x)-(b_j)_{2Q})_\sigma|^{r'}dx\right)^{\frac{1}{r'}} \\
&\times\left(\int_{2Q}|T(b_j-(b_j)_{2Q})_{\sigma^c}(f)(x)|^r dx\right)^{\frac{1}{r}} \\
\leq\ & C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|2Q|^{1+\frac{m\beta}{n}-\frac{1}{p}}}|2Q|^{\frac{\lambda_1\beta}{n}+\sum\frac{1}{r't_i}} \\
&\times\prod_i\frac{1}{|2Q|^\beta}\left(\frac{1}{|2Q|}\int_{2Q}|(b_j(x)-(b_j)_{2Q})_\sigma|^{r't_i}dx\right)^{\frac{1}{r't_i}} \\
&\times\left(\int_{2Q}|(b_j-(b_j)_{2Q})_{\sigma^c}(f)(x)|^r dx\right)^{\frac{1}{r}} \\
\leq\ & C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|2Q|^{1+\frac{m\beta}{n}-\frac{1}{p}}}|2Q|^{\frac{\lambda_1\beta}{n}+\sum\frac{1}{r't_i}}||b_\sigma||_{\dot\Lambda_\beta}\left(\int_{2Q}|(f)(x)|^{rt}dx\right)^{\frac{1}{rt}} \\
&\times\prod_k|2Q|^{\frac{\lambda_2\beta}{n}+\sum\frac{1}{r's_k}}\frac{1}{|2Q|^\beta}\left(\frac{1}{|2Q|}\int_{2Q}|(b_j-(b_j)_{2Q})_{\sigma^c}|^{r's_k}dx\right)^{\frac{1}{r's_k}} \\
\leq\ & C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|2Q|^{1+\frac{m\beta}{n}-\frac{1}{p}}}|2Q|^{\frac{\lambda_1\beta}{n}+\frac{1}{r'q}}||b_\sigma||_{\dot\Lambda_\beta}||f||_{L^p}|2Q|^{\frac{\lambda_2\beta}{n}+\frac{1}{rq}}||b_{\sigma^c}||_{\dot\Lambda_\beta} \\
\leq\ & C||\vec{b}||_{\dot\Lambda_\beta}||f||_{L^p}.
\end{aligned}
$$

For $I_4(x)$, set $\frac{1}{q_1}+\frac{1}{q_2}\cdots\frac{1}{q_m}+\frac{1}{q}=1$ with $1/p+1/q+1/t=1$, using Hölder's inequality and Lemma 3, we have

$$
\begin{aligned}
& |T(\prod_{j=1}^{m}(b_j-(b_j)_{2Q})f_2)(x)-T(\prod_{j=1}^{m}(b_j-(b_j)_{2Q})f_2)(x_0)| \\
= & \left|\int_{(2Q)^c}\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})f_2(y)(K(x,y)-K(x_0,y))dy\right| \\
\le & \int_{(2Q)^c}\prod_{j=1}^{m}|(b_j(y)-(b_j)_{2Q})||f(y)||(K(x,y)-K(x_0,y))|dy \\
\le & C\sum_{k=1}^{\infty}\left(\int_{2^k|x-x_0|\le|y-x_0|<2^{k+1}|x-x_0|}|K(x,y)-K(x_0,y)|^t dy\right)^{1/t} \\
& \times\prod_{j=1}^{m}\left(\int_{|y-x_0|<2^{k+1}|x-x_0|}|b_j(y)-(b_j)_{2Q}|^{q_j}dy\right)^{1/q_j} \\
& \times\left(\int_{|y-x_0|<2^{k+1}|x-x_0|}|f(y)|^p dy\right)^{1/p} \\
\le & C\sum_{k=1}^{\infty}C_k\frac{1}{(2^kd)^{\frac{n}{t'}}}\prod_{j=1}^{m}\left(\int_{|y-x_0|<2^{k+1}|x-x_0|}|b_j(y)-(b_j)_{2Q}|^{q_j}dy\right)^{1/q_j} \\
& \qquad\times\left(\int_{|y-x_0|<2^{k+1}|x-x_0|}|f(y)|^p dy\right)^{1/p} \\
\le & C\sum_{k=1}^{\infty}C_k\frac{1}{(2^kd)^{\frac{n}{t'}}}\prod_{j=1}^{m}\left(\int_{2^{k+1}Q}|b_j(y)-(b_j)_{2Q}|^{q_j}dy\right)^{1/q_j}\left(\int_{2^{k+1}Q}|f(y)|^p dy\right)^{1/p} \\
\le & C\sum_{k=1}^{\infty}C_k\frac{1}{(2^kd)^{\frac{n}{t'}}}\Bigg[\prod_{j=1}^{m}\left(\int_{2^{k+1}Q}|b_j(y)-(b_j)_{2^{k+1}Q}|^{q_j}dx\right)^{\frac{1}{q_j}} \\
& \qquad+\prod_{j=1}^{m}|(b_j)_{2^{k+1}Q}-(b_j)_{2Q}||2^{k+1}Q|^{\frac{1}{q_j}}\Bigg]||f||_{L^p} \\
\le & C\sum_{k=1}^{\infty}C_k\frac{1}{|2^{k+1}Q|^{\frac{1}{t'}}} \\
& \times\Bigg[\prod_{j=1}^{m}\frac{1}{|2^{k+1}Q|^{\frac{\beta}{n}}}\left(\frac{1}{2^{k+1}Q}\int_{2^{k+1}Q}|b_j(y)-(b_j)_{2^{k+1}Q}|^{q_j}dx\right)^{\frac{1}{q_j}}|2^{k+1}Q|^{\frac{\beta}{n}+\frac{1}{q_j}} \\
& \quad+\prod_{j=1}^{m}|(b_j)_{2^{k+1}Q}-(b_j)_{2Q}||2^{k+1}Q|^{\frac{1}{q_j}}\Bigg]||f||_{L^p}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\sum_{k=1}^{\infty} C_k \frac{1}{|2^{k+1}Q|^{\frac{1}{t'}}}\\
&\qquad \times\left[|2^{k+1}Q|^{\frac{m\beta}{n}+\frac{1}{q_1}+\frac{1}{q_2}\cdots\frac{1}{q_m}}||\vec{b}||_{\dot{\Lambda}_\beta}+Ck||\vec{b}||_{\dot{\Lambda}_\beta}|2^{k+1}Q|^{\frac{m\beta}{n}+\frac{1}{q_1}+\frac{1}{q_2}\cdots\frac{1}{q_m}}\right]||f||_{L^p}\\
&\leq C\sum_{k=1}^{\infty} C_k|2^{k+1}Q|^{\frac{m\beta}{n}-\frac{1}{p}}||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{L^p}\\
&\leq C|2^{k+1}Q|^{\frac{m\beta}{n}-\frac{1}{p}}||b_1||_{\dot{\Lambda}_\beta}||f||_{L^p}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\frac{1}{|2Q|^{1+\frac{m\beta}{n}-\frac{1}{p}}}\int_{2Q} I_4(x)dx\\
\leq\ & C\frac{1}{|2Q|^{1+\frac{m\beta}{n}-\frac{1}{p}}}|2Q|^{\frac{m\beta}{n}-\frac{1}{p}}||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{L^p}|2Q|\leq C||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{L^p}.
\end{aligned}
$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Fix a ball $Q=Q(0,l)$, there exists $\epsilon_0\in\mathbf{Z}$ such that $2^{\epsilon_0-1}\leq l<2^{\epsilon_0}$. We choose x_0 such that $2l<d(x_0,x)<3l$. It is only to prove that

$$
|Q_{\epsilon_0}|^{\alpha+\frac{1}{q_2}}\left(\frac{1}{|Q_{\epsilon_0}|}\int_{Q_{\epsilon_0}}|T_{\vec{b}}(f)(x)-T_{\vec{b}}(f_2)(x_0)|^{q_2}dx\right)^{\frac{1}{q_2}}\leq C||f||_{\dot{K}^{\alpha,\infty}_{q_1}}.
$$

We write $f_1=f\chi_{4Q_{\epsilon_0}}$ and $f_2=f\chi_{X\backslash 4Q_{\epsilon_0}}$, then

$$
\begin{aligned}
&|T_{\vec{b}}(f)(x)-T_{\vec{b}}(f_2)(x_0)|\\
\leq\ & |T_{\vec{b}}(f_1)(x)|+|T_{\vec{b}}(f_2)(x)-T_{\vec{b}}(f_2)(x_0)|\\
\leq\ & |T_{\vec{b}}(f_1)(x)|+|T(\prod_{j=1}^{m}(b_j-(b_j)_Q)f_2)(x)-T(\prod_{j=1}^{m}(b_j-(b_j)_Q)f_2)(x_0)|\\
&+|\prod_{j=1}^{m}(b_j(x)-(b_j)_Q)||T(f_2)(x)-T(f_2)(x_0)|.
\end{aligned}
$$

So

$$\begin{aligned}
& |Q_{\epsilon_0}|^{\alpha+\frac{1}{q_2}}\left(\frac{1}{|Q_{\epsilon_0}|}\int_{Q_{\epsilon_0}}|T_{\vec{b}}(f)(x)-T_{\vec{b}}(f_2)(x_0)|^{q_2}dx\right)^{\frac{1}{q_2}} \\
\leq\ & |Q_{\epsilon_0}|^{\alpha+\frac{1}{q_2}}\left(\frac{1}{|Q_{\epsilon_0}|}\int_{Q_{\epsilon_0}}|T_{\vec{b}}(f_1)(x)|^{q_2}dx\right)^{\frac{1}{q_2}} \\
& +|Q_{\epsilon_0}|^{\alpha+\frac{1}{q_2}}\left(\frac{1}{|Q_{\epsilon_0}|}\int_{Q_{\epsilon_0}}|T(\prod_{j=1}^{m}(b_j-(b_j)_Q)f_2)(x)-T(\prod_{j=1}^{m}(b_j-(b_j)_Q)f_2)(x_0)|^{q_2}dx\right)^{\frac{1}{q_2}} \\
& +|Q_{\epsilon_0}|^{\alpha+\frac{1}{q_2}}\left(\frac{1}{|Q_{\epsilon_0}|}\int_{Q_{\epsilon_0}}|\prod_{j=1}^{m}(b_j(x)-(b_j)_Q)||T(f_2)(x)-T(f_2)(x_0)|^{q_2}dx\right)^{\frac{1}{q_2}} \\
=\ & W_1+W_2+W_3.
\end{aligned}$$

For W_1, by Lemma 2 and Lemma 4, we get

$$\begin{aligned}
W_1 &\leq C|Q_{\epsilon_0}|^{\alpha+\frac{1}{q_2}-\frac{1}{q_2}}\left(\int_{Q_{\epsilon_0}}|f_1(x)|^{q_1}dX\right)^{\frac{1}{q_1}} \\
&\leq C|Q_{\epsilon_0}|^{\alpha}||f\chi_{Q_{\epsilon_0}}||_{L^{q_1}} \\
&\leq C||f||_{\dot{K}^{\alpha,\infty}_{q_1}}.
\end{aligned}$$

For W_2, similar to the estimates of $I_4(x)$in Theorem 1, let $\frac{1}{p_1}+\cdots+\frac{1}{p_m}+\frac{1}{q_1}+\frac{1}{t}=1$, by Hölder's inequality and the Minkowski's inequality, we obtain

$$\begin{aligned}
& |T(\prod_{j=1}^{m}(b_j-(b_j)_{2Q})f_2)(x)-T(\prod_{j=1}^{m}(b_j-(b_j)_{2Q})f_2)(x_0)| \\
\leq\ & C\sum_{k=1}^{\infty}\int_{Q_{\epsilon_0+k}}|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2B})||f(y)||(K(x,y)-K(x_0,y))|dy \\
\leq\ & C\sum_{k=1}^{\infty}\left(\int_{Q_{\epsilon_0+k}\setminus Q_{\epsilon_0+k-1}}|K(x,y)-K(x_0,y)|^t dy\right)^{1/t} \\
& \times\prod_{j=1}^{m}\left(\int_{Q_{\epsilon_0+k}\setminus Q_{\epsilon_0+k-1}}|b_j(y)-(b_j)_{2Q}|^{p_j}dy\right)^{1/p_j}\left(\int_{Q_{\epsilon_0+k}\setminus Q_{\epsilon_0+k-1}}|f(y)|^{q_1}dy\right)^{1/q_1}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{k=1}^{\infty}C_k\frac{1}{|Q_{\epsilon_0+k}|^{\frac{1}{t'}}}\prod_{j=1}^{m}\left(\int_{Q_{\epsilon_0+k}\setminus Q_{\epsilon_0+k-1}}|b_j(y)-(b_j)_{2Q}|^{p_j}dy\right)^{1/p_j}\\
&\qquad\times\left(\int_{Q_{\epsilon_0+k}\setminus Q_{\epsilon_0+k-1}}|f(y)|^{q_1}dy\right)^{1/q_1}\\
&\leq C\sum_{k=1}^{\infty}C_k\frac{1}{|Q_{\epsilon_0+k}|^{\frac{1}{t'}}}\left[\prod_{j=1}^{m}\left(\int_{Q_{\epsilon_0+k}}|b_j(y)-(b_j)_{2^{k+1}Q}|^{p_j}dx\right)^{\frac{1}{p_j}}\right.\\
&\quad\left.+\prod_{j=1}^{m}|(b_j)_{2^{k+1}Q}-(b_j)_{2Q}||2^{k+1}Q|^{\frac{1}{p_j}}\right]||f\chi_{Q_{\epsilon_0}}||_{L^{q_1}}\\
&\leq C\sum_{k=1}^{\infty}C_k\frac{1}{|Q_{\epsilon_0+k}|^{\frac{1}{t'}}}\left[|Q_{\epsilon_0+k}|^{\frac{m\beta}{n}+\frac{1}{p_1}+\frac{1}{p_2}\cdots\frac{1}{p_m}}||\vec{b}||_{\dot{\Lambda}_\beta}\right.\\
&\qquad\left.+C_k||\vec{b}||_{\dot{\Lambda}_\beta}|Q_{\epsilon_0+k}|^{\frac{m\beta}{n}+\frac{1}{p_1}+\frac{1}{p_2}\cdots\frac{1}{p_m}}\right]||f\chi_{Q_{\epsilon_0}}||_{L^{q_1}}\\
&\leq C\sum_{k=1}^{\infty}C_k|Q_{\epsilon_0+k}|^{\frac{m\beta}{n}-\frac{1}{q_1}-\alpha}||\vec{b}||_{\dot{\Lambda}_\beta}|Q_{\epsilon_0+k}|^{\alpha}||f\chi_{Q_{\epsilon_0}}||_{L^{q_1}}\\
&\leq C|Q_{\epsilon_0+k}|^{-\frac{1}{q_2}-\alpha}||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{\dot{K}^{\alpha,\infty}_{q_1}},
\end{aligned}
$$

thus

$$
\begin{aligned}
W_2 &\leq C|Q_{\epsilon_0}|^{\alpha+\frac{1}{q_2}}|Q_{\epsilon_0+k}|^{-\frac{1}{q_2}-\alpha}||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{\dot{K}^{\alpha,\infty}_{q_1}}\\
&\leq C||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{\dot{K}^{\alpha,\infty}_{q_1}}.
\end{aligned}
$$

For W_3, with the same method as above, let $\frac{1}{d_1}+\cdots+\frac{1}{d_m}=1$, using Lemma 2, we have

$$
\begin{aligned}
|T(f_2)(x)-T(f_2)(x_0)| &\leq \int_{Q_{\epsilon_0}}|K(x,y)-K(x_0,y)||f(y)|dy\\
&\leq C\frac{1}{|Q_{\epsilon_0}|}||f\chi_{Q_{\epsilon_0}}||_{L^{q_1}}|Q_{\epsilon_0}|^{1-\frac{1}{q_1}}\\
&\leq C|Q_{\epsilon_0}|^{-\frac{1}{q_1}-\alpha}|Q_{\epsilon_0}|^{\alpha}||f\chi_{Q_{\epsilon_0}}||_{L^{q_1}}\\
&\leq C|Q_{\epsilon_0}|^{-\frac{1}{q_1}-\alpha}||f||_{\dot{K}^{\alpha,\infty}_{q_1}},
\end{aligned}
$$

thus

$$\begin{aligned} W_3 &\leq |Q_{\epsilon_0}|^{\alpha+\frac{1}{q_2}}|Q_{\epsilon_0}|^{-\frac{1}{q_1}-\alpha}||f||_{\dot{K}^{\alpha,\infty}_{q_1}}|Q_{\epsilon_0}|^{-\frac{1}{q_2}}\left(\int_{Q_{\epsilon_0}}|\prod_{j=1}^{m}(b_j(x)-(b_j)_{2Q})|^{q_2}dx\right)^{\frac{1}{q_2}} \\ &\leq C|Q_{\epsilon_0}|^{-\frac{1}{q_1}}||f||_{\dot{K}^{\alpha,\infty}_{q_1}}|Q_{\epsilon_0}|^{\frac{m\beta}{n}+\frac{1}{q_2}} \\ &\quad\times\prod_{j=1}^{m}\frac{1}{|Q_{\epsilon_0}|^{\frac{\beta}{n}}}\left(\frac{1}{|Q_{\epsilon_0}|}\int_{Q_{\epsilon_0}}|(b_j(x)-(b_j)_{2Q})|^{q_2d_j}dx\right)^{\frac{1}{q_2d_j}} \\ &\leq C||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{\dot{K}^{\alpha,\infty}_{q_1}}. \end{aligned}$$

This completes the proof of Theorem 2.

References

[1] **D. C. Chang, J. F. Li, and J. Xiao**, Weighted scale estimates for Calderón-Zygmund type operators, *Contemporary Mathematics*, **446 (2007)**, 61-70.

[2] **W. G. Chen**, Besov estimates for a class of multilinear singular integrals, *Acta Math. Sinica*, **16 (2000)**, 613-626.

[3] **R. R. Coifman, R. Rochberg, and G. Weiss**, Fractorization theorems for Hardy spaces in several variables, *Ann. of Math.*, **103(1976)**, 611-635.

[4] **J. Garcia-Cuerva and J. L. Rubio de Francia**, *Weighted norm inequalities and related topics*, North-Holland Math. 116, Amsterdam, 1985.

[5] **S. Guo, C. X. Huang, and L. Z. Liu**, Boundedness for multilinear commutator of singular integral operator with weighted Lipschitz functions, to appear.

[6] **S. Janson**, Mean oscillation and commutators of singular integral operators, *Ark. Math.*, **16 (1978)**, 263-270.

[7] **Y. Lin**, Sharp maximal function estimates for Calderón-Zygmund type operators and commutators, *Acta Math. Scientia*, **31 (A)**, (2011), 206-215.

[8] **M. Paluszynski**, Characterization of the Besov spaces via the commutator operator of Coifman, **44 (1995)**, 1-17.

[9] **C. Pérez**, Endpoint estimate for commutators of singular integral operators, *J. Func. Anal.*, **128 (1995)**, 163-185.

[10] **C. Pérez and G. Pradolini**, Sharp weighted endpoint estimates for commutators of singular integral operators, *Michigan Math. J.*, **49 (2001)**, 23-37.

[11] **Pérez C. and R. Trujillo-Gonzalez**, Sharp Weighted estimates for multilinear commutators, *J. London Math. Soc.*, **65 (2002)**, 672-692.

[12] **E. M. Stein**, *Harmonic analysis: real variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, Princeton NJ, 1993.

[13] **D. C. Yang**, The central Campanato spaces and its applications, *Approx. Theory and Appl.*, **10 (4)**, (1994), 85-99.

Sheng Guo

Chuangxia Huang

Lanzhe Liu

Department of Mathematics
Changsha University of Science and Technology
Changsha, 410077
P. R. of China

E-mail: lanzheliu@163.com

Received: 4.06.2012

Accepted: 19.10.2012