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Sharp Function Estimates and Boundedness for Toeplitz Type Operators Associated to General Integral Operator

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Abstract. In this paper, we establish the sharp maximal function estimates for the Toeplitz type operators associated to some integral operator with general kernel and the Lipschitz functions. As an application, we obtain the boundedness of the Toeplitz type operators on the Lebesgue, Morrey and Triebel-Lizorkin space. The operator includes the Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

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1. Introduction and Preliminaries

As the development of the singular integral operators (see [6][22][23]), their commutators have been well studied. In [3][20][21], the authors prove that the commutators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [2]) proves a similar result when singular integral operators are replaced by the fractional integral operators. In [7][17], the boundedness for the commutators generated by the singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(R^n)(1 < p < \infty)$ spaces are obtained. In [1], some singular integral operators with general kernel are introduced, and the boundedness for the operators and their commutators generated by BMO and Lipschitz functions are obtained (see [1][10]). In [8][9], some Toeplitz type operators related to the singular integral operators and strongly singular integral operators are introduced, and the boundedness for the operators generated by BMO and Lipschitz functions are obtained. In this paper, we will study the Toeplitz type operators generated by some integral operators with general kernel and the Lipschitz functions. As an application, we obtain the boundedness of the operators on the Lebesgue, Morrey and Triebel-Lizorkin space. The operator includes Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$M^{\#}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

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where, and in what follows, $f_Q = |Q|^{-1}\int_Q f(x)dx$. It is well-known that (see [6][22])

$$M^{\#}(f)(x) \approx \sup_{Q\ni x}\inf_{c\in C}\frac{1}{|Q|}\int_Q |f(y)-c|dy.$$

Let

$$M(f)(x) = \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(y)|dy.$$

For $\eta > 0$, let $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$.

For $0 < \eta < 1$ and $1 \le r < \infty$, set

$$M_{\eta,r}(f)(x) = \sup_{Q\ni x}\left(\frac{1}{|Q|^{1-r\eta/n}}\int_Q |f(y)|^r dy\right)^{1/r}.$$

The A_p weight is defined by (see [6])

$$A_p = \left\{w \in L^1_{loc}(R^n) : \sup_Q\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} < \infty\right\},$$

$1 < p < \infty$, and

$$A_1 = \{w \in L^p_{loc}(R^n) : M(w)(x) \le Cw(x), a.e.\}.$$

For $\beta > 0$ and $p > 1$, let $\dot{F}^{\beta,\infty}_p(R^n)$ be the homogeneous Triebel-Lizorkin space (see [17]).

For $\beta > 0$, the Lipschitz space $Lip_\beta(R^n)$ is the space of functions f such that

$$||f||_{Lip_\beta} = \sup_{\substack{x,y\in R^n\\ x\neq y}}\frac{|f(x)-f(y)|}{|x-y|^\beta} < \infty.$$

Definition 1.1. *Let φ be a positive, increasing function on R^+ and there exists a constant $D > 0$ such that*

$$\varphi(2t) \le D\varphi(t) \quad for \quad t \ge 0.$$

Let f be a locally integrable function on R^n. Set, for $1 \le p < \infty$,

$$||f||_{L^{p,\varphi}} = \sup_{x\in R^n,\ d>0}\left(\frac{1}{\varphi(d)}\int_{Q(x,d)}|f(y)|^p dy\right)^{1/p},$$

where $Q(x,d) = \{y \in R^n : |x-y| < d\}$. The generalized Morrey space is defined by

$$L^{p,\varphi}(R^n) = \{f \in L^1_{loc}(R^n) : ||f||_{L^{p,\varphi}} < \infty\}.$$

If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(R^n) = L^{p,\delta}(R^n)$, which is the classical Morrey spaces (see [18][19]). If $\varphi(d) = 1$, then $L^{p,\varphi}(R^n) = L^p(R^n)$, which is the Lebesgue spaces (see [4]).

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [4][5][11][16]).

In this paper, we will study some integral operators as following (see [1]).

Definition 1.2. *Let $F_t(x,y)$ be defined on $R^n \times R^n \times [0,+\infty)$ and b be a locally integrable function on R^n, set*

$$F_t(f)(x) = \int_{R^n} F_t(x,y)f(y)dy$$

for every bounded and compactly supported function f. And F_t satisfies: there is a sequence of positive constant numbers $\{C_j\}$ such that for any $j \geq 1$,

$$\int_{2|y-z|<|x-y|} (||F_t(x,y) - F_t(x,z)|| + ||F_t(y,x) - F_t(z,x)||)dx \leq C,$$

and

$$\left(\int_{2^j|z-y|\leq|x-y|<2^{j+1}|z-y|} (||F_t(x,y) - F_t(x,z)|| + ||F_t(y,x) - F_t(z,x)||)^q dy\right)^{1/q}$$
$$\leq C_j(2^j|z-y|)^{-n/q'},$$

where $1 < q' < 2$ and $1/q + 1/q' = 1$.

Let H be the Banach space $H = \{h : ||h|| < \infty\}$. For each fixed $x \in R^n$, we view $F_t(f)(x)$ as the mapping from $[0, +\infty)$ to H. Set

$$T(f)(x) = ||F_t(f)(x)||,$$

which T is bounded on $L^2(R^n)$. The Toeplitz type operator related to T is defined by

$$T^b(f) = ||F_t^b(f)||,$$

where

$$F_t^b(f) = \sum_{k=1}^m F_t^{k,1} M_b F_t^{k,2}(f),$$

moreover, $F_t^{k,1}(f)$ are $F_t(f)$ or $\pm I$(the identity operator), $T^{k,2}(f) = ||F_t^{k,2}(f)||$ are the bounded linear operators on $L^p(R^n)$ for $1 < p < \infty$ and $k = 1, ..., m$, $M_b(f) = bf$.

Note that the commutator $[b, T](f) = bT(f) - T(bf)$ is a particular operator of the Toeplitz type operators T^b. The Toeplitz type operators T^b are the non-trivial generalizations of the commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [20][21]). The main purpose of this paper is to prove the sharp maximal inequalities for the Toeplitz type operators T^b. As the application, we obtain the the L^p-norm inequality and Triebel-Lizorkin spaces boundedness for the Toeplitz type operators T^b.

2. Theorems

We shall prove the following theorems.

Theorem 2.1. *Let T be the integral operator as Definition 1.2, the sequence $\{C_j\} \in l^1$, $0 < \beta < 1$, $q' \leq s < \infty$ and $b \in Lip_\beta(R^n)$. If $g \in L^u(R^n)(1 < u < \infty)$ and $F_t^1(g) = 0$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

$$M^\#(T^b(f))(\tilde{x}) \leq C||b||_{Lip_\beta} \sum_{k=1}^m M_{\beta,s}(T^{k,2}(f))(\tilde{x}).$$

Theorem 2.2. *Let T be the integral operator as Definition 1.2, the sequence $\{2^{j\beta}C_j\} \in l^1$, $0 < \beta < 1$, $q' \leq s < \infty$ and $b \in Lip_\beta(R^n)$. If $g \in L^u(R^n)(1 < u < \infty)$ and $F_t^1(g) = 0$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

$$\sup_{Q \ni \tilde{x}} \frac{1}{|Q|^{1+\beta/n}} \int_Q \left|T^b(f)(x) - C_0\right| dx \leq C||b||_{Lip_\beta} \sum_{k=1}^m M_s(T^{k,2}(f))(\tilde{x}).$$

Theorem 2.3. *Let T be the integral operator as Definition 1.2, the sequence $\{jC_j\} \in l^1$, $0 < \beta < 1$, $q' \le s < \infty$ and $b \in BMO(R^n)$. If $g \in L^u(R^n)(1 < u < \infty)$ and $F_t^1(g) = 0$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

$$M^{\#}(T^b(f))(\tilde{x}) \le C||b||_{BMO} \sum_{k=1}^{m} M_s(T^{k,2}(f))(\tilde{x}).$$

Theorem 2.4. *Let T be the integral operator as Definition 1.2, the sequence $\{C_j\} \in l^1$, $0 < \beta < \min(1, n/q')$, $q' < p < n/\beta$, $1/r = 1/p - \beta/n$ and $b \in Lip_\beta(R^n)$. If $g \in L^u(R^n)(1 < u < \infty)$ and $F_t^1(g) = 0$, then T^b is bounded from $L^p(R^n)$ to $L^r(R^n)$.*

Theorem 2.5. *Let T be the integral operator as Definition 1.2, the sequence $\{C_j\} \in l^1$, $0 < \beta < \min(1, n/q')$, $q' < p < n/\beta$, $0 < D < 2^n$ and $b \in Lip_\beta(R^n)$. If $g \in L^u(R^n)(1 < u < \infty)$ and $F_t^1(g) = 0$, then T^b is bounded from $L^{p,\varphi}(R^n)$ to $L^{r,\varphi}(R^n)$.*

Theorem 2.6. *Let T be the integral operator as Definition 1.2, the sequence $\{2^{j\beta}C_j\} \in l^1$, $0 < \beta < \min(1, n/q')$, $q' < p < n/\beta$ and $b \in Lip_\beta(R^n)$. If $g \in L^u(R^n)(1 < u < \infty)$ and $F_t^1(g) = 0$, then T^b is bounded from $L^p(R^n)$ to $\dot{F}_p^{\beta,\infty}(R^n)$.*

Theorem 2.7. *Let T be the integral operator as Definition 1.2, the sequence $\{jC_j\} \in l^1$, $q' < p < n/\beta$ and $b \in BMO(R^n)$. If $g \in L^u(R^n)(1 < u < \infty)$ and $F_t^1(g) = 0$, then T^b is bounded on $L^p(R^n)$.*

Theorem 2.8. *Let T be the integral operator as Definition 1.2, the sequence $\{jC_j\} \in l^1$, $q' < p < n/\beta$, $0 < D < 2^n$ and $b \in BMO(R^n)$. If $g \in L^u(R^n)(1 < u < \infty)$ and $F_t^1(g) = 0$, then T^b is bounded on $L^{p,\varphi}(R^n)$.*

3. Proofs of Theorems

To prove the theorems, we need the following lemmas.

Lemma 3.1. *(see [1]) Let T be the integral operator as Definition 1.2. Then T is bounded on $L^p(R^n)$ for $1 < p < \infty$.*

Lemma 3.2. *(see [17]) For $0 < \beta < 1$ and $1 < p < \infty$, we have*

$$||f||_{\dot{F}_p^{\beta,\infty}} \approx \left|\left| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q| dx \right|\right|_{L^p}$$
$$\approx \left|\left| \sup_{Q \ni \cdot} \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - c| dx \right|\right|_{L^p}.$$

Lemma 3.3. *(see [6]) Let $0 < p < \infty$ and $w \in \cup_{1 \le r < \infty} A_r$. Then, for any smooth function f for which the left-hand side is finite,*

$$\int_{R^n} M(f)(x)^p w(x) dx \le C \int_{R^n} M^{\#}(f)(x)^p w(x) dx.$$

Lemma 3.4. *(see [2]) Suppose that $0 < \eta < n$, $1 \le s < p < n/\eta$ and $1/q = 1/p - \eta/n$. Then*

$$||M_{\eta,s}(f)||_{L^q} \le C||f||_{L^p}.$$

Lemma 3.5. *Let $1 < p < \infty$, $0 < D < 2^n$. Then, for any smooth function f for which the left-hand side is finite,*

$$||M(f)||_{L^{p,\varphi}} \le C||M^{\#}(f)||_{L^{p,\varphi}}.$$

Proof. For any cube $Q = Q(x_0, d)$ in R^n, we know $M(\chi_Q) \in A_1$ for any cube $Q = Q(x, d)$ by [6]. Noticing that $M(\chi_Q) \leq 1$ and $M(\chi_Q)(x) \leq d^n/(|x - x_0| - d)^n$ if $x \in Q^c$, by Lemma 3.3, we have, for $f \in L^{p,\varphi}(R^n)$,

$$
\begin{aligned}
&\int_Q M(f)(x)^p dx = \int_{R^n} M(f)(x)^p \chi_Q(x) dx \\
\leq\ & \int_{R^n} M(f)(x)^p M(\chi_Q)(x) dx \leq C \int_{R^n} M^{\#}(f)(x)|^p M(\chi_Q)(x) dx \\
=\ & C\left(\int_Q M^{\#}(f)(x)^p M(\chi_Q)(x) dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q\setminus 2^k Q} M^{\#}(f)(x)^p M(\chi_Q)(x) dx\right) \\
\leq\ & C\left(\int_Q M^{\#}(f)(x)^p dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q\setminus 2^k Q} M^{\#}(f)(x)^p \frac{|Q|}{|2^{k+1}Q|} dx\right) \\
\leq\ & C\left(\int_Q M^{\#}(f)(x)^p dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q} M^{\#}(f)(x)^p 2^{-kn} dy\right) \\
\leq\ & C||M^{\#}(f)||^p_{L^{p,\varphi}} \sum_{k=0}^{\infty} 2^{-kn}\varphi(2^{k+1}d) \\
\leq\ & C||M^{\#}(f)||^p_{L^{p,\varphi}} \sum_{k=0}^{\infty} (2^{-n}D)^k \varphi(d) \\
\leq\ & C||M^{\#}(f)||^p_{L^{p,\varphi}} \varphi(d),
\end{aligned}
$$

thus

$$\left(\frac{1}{\varphi(d)}\int_Q M(f)(x)^p dx\right)^{1/p} \leq C\left(\frac{1}{\varphi(d)}\int_Q M^{\#}(f)(x)^p dx\right)^{1/p}$$

and

$$||M(f)||_{L^{p,\varphi}} \leq C||M^{\#}(f)||_{L^{p,\varphi}}.$$

This finishes the proof. □

Lemma 3.6. *Let $0 < D < 2^n$, $1 \leq s < p < n/\eta$ and $1/r = 1/p - \eta/n$. Then*

$$||M_{\eta,s}(f)||_{L^{r,\varphi}} \leq C||f||_{L^{p,\varphi}}.$$

The proof of the Lemma is similar to that of Lemma 3.5 by Lemma 3.4, we omit the details.

Proof. Of Theorem 2.1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - C_0\right| dx \leq C||b||_{Lip_\beta} \sum_{k=1}^{m} M_{\beta,s}(T^{k,2}(f))(\tilde{x}).$$

Without loss of generality, we may assume $T^{k,1}$ are $T(k = 1, ..., m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$,

$$F_t^b(f)(x) = F_t^{b-b_Q}(f)(x) = F_t^{(b-b_Q)\chi_{2Q}}(f)(x) + F_t^{(b-b_Q)\chi_{(2Q)^c}}(f)(x) = f_1(x) + f_2(x).$$

Then

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - ||f_2(x_0)||\right| dx = \frac{1}{|Q|}\int_Q |||F_t^b(f)(x)|| - ||f_2(x_0)|||dx \\
\leq\ & \frac{1}{|Q|}\int_Q ||F_t^b(f)(x) - f_2(x_0)||dx \\
\leq\ & \frac{1}{|Q|}\int_Q ||f_1(x)||dx + \frac{1}{|Q|}\int_Q ||f_2(x) - f_2(x_0)||dx = I_1 + I_2.
\end{aligned}$$

For I_1, by Hölder's inequality and Lemma 3.1, we obtain

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q ||F_t^{k,1} M_{(b-b_Q)\chi_{2Q}} F_t^{k,2}(f)(x)||dx \\
=\ & \frac{1}{|Q|}\int_Q |T^{k,1} M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)|dx \\
\leq\ & \left(\frac{1}{|Q|}\int_{R^n} |T^{k,1} M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)|^s dx\right)^{1/s} \\
\leq\ & C|Q|^{-1/s}\left(\int_{2Q} |M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)|^s dx\right)^{1/s} \\
\leq\ & C|Q|^{-1/s}\left(\int_{2Q} (|b(x) - b_Q||T^{k,2}(f)(x)|)^s dx\right)^{1/s} \\
\leq\ & C|Q|^{-1/s}||b||_{Lip_\beta}|2Q|^{\beta/n}|Q|^{1/s-\beta/n}\left(\frac{1}{|Q|^{1-s\beta/n}}\int_Q |T^{k,2}(f)(x)|^s dx\right)^{1/s} \\
\leq\ & C||b||_{Lip_\beta} M_{\beta,s}(T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

thus

$$\begin{aligned}
I_1 &\leq \sum_{k=1}^m \frac{1}{|Q|}\int_Q ||F_t^{k,1} M_{(b-b_Q)\chi_{2Q}} F_t^{k,2}(f)(x)||dx \\
&\leq C||b||_{Lip_\beta}\sum_{k=1}^m M_{\beta,s}(T^{k,2}(f))(\tilde{x}).
\end{aligned}$$

For I_2, by the boundedness of T and recalling that $s > q'$, we get, for $x \in Q$,

$$\begin{aligned}
&||F_t^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} F_t^{k,2}(f)(x) - F_t^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} F_t^{k,2}(f)(x_0)|| \\
\leq\ & \int_{(2Q)^c} |b(y) - b_{2Q}|||F_t(x,y) - F_t(x_0,y)|||T^{k,2}(f)(y)|dy \\
\leq\ & \sum_{j=1}^\infty \int_{2^j d\leq|y-x_0|<2^{j+1}d} ||F_t(x,y) - F_t(x_0,y)|||b(y) - b_{2Q}||T^{k,2}(f)(y)|dy \\
\leq\ & C||b||_{Lip_\beta}\sum_{j=1}^\infty |2^{j+1}Q|^{\beta/n}\left(\int_{2^j d\leq|y-x_0|<2^{j+1}d} ||F_t(x,y) - F_t(x_0,y)||^q dy\right)^{1/q} \\
& \times\left(\int_{2^{j+1}Q} |T^{k,2}(f)(y)|^{q'} dy\right)^{1/q'}
\end{aligned}$$

$$
\begin{aligned}
&\leq C||b||_{Lip_\beta}\sum_{j=1}^{\infty}|2^{j+1}Q|^{\beta/n}C_j(2^jd)^{-n/q'}|2^{j+1}Q|^{1/q'-\beta/n}\\
&\quad\times\left(\frac{1}{|2^{j+1}Q|^{1-s\beta/n}}\int_{2^{j+1}Q}|T^{k,2}(f)(y)|^s dy\right)^{1/s}\\
&\leq C||b||_{Lip_\beta}M_{\beta,s}(T^{k,2}(f))(\tilde{x})\sum_{j=1}^{\infty}C_j\\
&\leq C||b||_{Lip_\beta}M_{\beta,s}(T^{k,2}(f))(\tilde{x}),
\end{aligned}
$$

thus

$$
\begin{aligned}
I_2 &\leq \frac{1}{|Q|}\int_Q\sum_{k=1}^{m}||F_t^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}F_t^{k,2}(f)(x)-F_t^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}F_t^{k,2}(f)(x_0)||dx\\
&\leq C||b||_{Lip_\beta}\sum_{k=1}^{m}M_{\beta,s}(T^{k,2}(f))(\tilde{x}).
\end{aligned}
$$

These complete the proof of Theorem 2.1. □

Proof. Of Theorem 2.2. It suffices to prove for $f\in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$
\frac{1}{|Q|^{1+\beta/n}}\int_Q\left|T^b(f)(x)-C_0\right|dx\leq C||b||_{Lip_\beta}\sum_{k=1}^{m}M_s(T^{k,2}(f))(\tilde{x}).
$$

Without loss of generality, we may assume $T^{k,1}$ are $T(k=1,...,m)$. Fix a cube $Q=Q(x_0,d)$ and $\tilde{x}\in Q$. For $f_1=f\chi_{2Q}$ and $f_2=f\chi_{(2Q)^c}$, write

$$
F_t^b(f)(x)=F_t^{b-b_Q}(f)(x)=F_t^{(b-b_Q)\chi_{2Q}}(f)(x)+F_t^{(b-b_Q)\chi_{(2Q)^c}}(f)(x)=f_1(x)+f_2(x)
$$

and

$$
\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}}\int_Q\left|T^b(f)(x)-||f_2(x_0)||\right|dx=\frac{1}{|Q|^{1+\beta/n}}\int_Q|||F_t^b(f)(x)||-||f_2(x_0)|||dx\\
\leq&\ \frac{1}{|Q|^{1+\beta/n}}\int_Q||F_t^b(f)(x)-f_2(x_0)||dx\leq\frac{1}{|Q|^{1+\beta/n}}\int_Q||f_1(x)||dx\\
&+\frac{1}{|Q|^{1+\beta/n}}\int_Q||f_2(x)-f_2(x_0)||dx=I_3+I_4.
\end{aligned}
$$

By using the same argument as in the proof of Theorem 2.1, we get

$$
\begin{aligned}
I_3 &\leq \sum_{k=1}^{m}\frac{C}{|Q|^{\beta/n}}||b||_{Lip_\beta}|2Q|^{\beta/n}|Q|^{-1/s}\left(\int_{2Q}|T^{k,2}(f)(x)|^s dx\right)^{1/s}\\
&\leq C||b||_{Lip_\beta}\sum_{k=1}^{m}\left(\frac{1}{|2Q|}\int_{2Q}|T^{k,2}(f)(x)|^s dx\right)^{1/s}\\
&\leq C||b||_{Lip_\beta}\sum_{k=1}^{m}M_s(T^{k,2}(f))(\tilde{x}),
\end{aligned}
$$

$$\begin{aligned}
I_4 &\leq \sum_{k=1}^{m} \frac{1}{|Q|^{1+\beta/n}} \int_Q \sum_{j=1}^{\infty} \int_{2^j d \leq |y-x_0| < 2^{j+1} d} |b(y) - b_{2Q}|| \\
&\quad \times |F_t(x,y) - F_t(x_0,y)|||T^{k,2}(f)(y)|dydx \\
&\leq \sum_{k=1}^{m} \frac{C}{|Q|^{1+\beta/n}} \int_Q \sum_{j=1}^{\infty} ||b||_{Lip_\beta} |2^{j+1}Q|^{\beta/n} \\
&\quad \times \left(\int_{2^j d \leq |y-x_0| < 2^{j+1} d} ||F_t(x,y) - F_t(x_0,y)||^q dy \right)^{1/q} \\
&\quad \times \left(\int_{2^{j+1}Q} |T^{k,2}(f)(y)|^{q'} dy \right)^{1/q'} dx \\
&\leq C||b||_{Lip_\beta} \sum_{k=1}^{m} |Q|^{-\beta/n} \sum_{j=1}^{\infty} |2^{j+1}Q|^{\beta/n} C_j (2^j d)^{-n/q'} |2^{j+1}Q|^{1/q'} \\
&\quad \times \left(\frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} |T^{k,2}(f)(y)|^s dy \right)^{1/s} \\
&\leq C||b||_{Lip_\beta} \sum_{k=1}^{m} M_s(T^{k,2}(f))(\tilde{x}) \sum_{j=1}^{\infty} 2^{j\beta} C_j \\
&\leq C||b||_{Lip_\beta} \sum_{k=1}^{m} M_s(T^{k,2}(f))(\tilde{x}).
\end{aligned}$$

This completes the proof of Theorem 2.2. $\square$

Proof. Of Theorem 2.3. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|} \int_Q \left| T^b(f)(x) - C_0 \right| dx \leq C||b||_{BMO} \sum_{k=1}^{m} M_s(T^{k,2}(f))(\tilde{x}).$$

Without loss of generality, we may assume $T^{k,1}$ are $T (k = 1, ..., m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. For $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$, similar to the proof of Theorem 1, we have

$$F_t^b(f)(x) = F_t^{b-b_Q}(f)(x) = F_t^{(b-b_Q)\chi_{2Q}}(f)(x) + F_t^{(b-b_Q)\chi_{(2Q)^c}}(f)(x) = f_1(x) + f_2(x)$$

and

$$\begin{aligned}
&\frac{1}{|Q|} \int_Q \left| T^b(f)(x) - ||f_2(x_0)|| \right| dx = \frac{1}{|Q|} \int_Q |||F_t^b(f)(x)|| - ||f_2(x_0)|||dx \\
\leq &\frac{1}{|Q|} \int_Q ||F_t^b(f)(x) - f_2(x_0)||dx \leq \frac{1}{|Q|} \int_Q ||f_1(x)||dx \\
&+ \frac{1}{|Q|} \int_Q ||f_2(x) - f_2(x_0)||dx = I_5 + I_6.
\end{aligned}$$

For I_5, choose $1 < r < s$, by Hölder's inequality and the boundedness of T, we obtain

$$\begin{aligned}
& \frac{1}{|Q|}\int_Q ||F_t^{k,1}M_{(b-b_Q)\chi_{2Q}}F_t^{k,2}(f)(x)||dx \\
\leq & \frac{1}{|Q|}\int_Q |T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)|dx \\
\leq & \left(\frac{1}{|Q|}\int_{R^n} |T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)|^r dx\right)^{1/r} \\
\leq & C|Q|^{-1/r}\left(\int_{R^n} |M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)|^r dx\right)^{1/r} \\
\leq & C|Q|^{-1/r}\left(\int_{2Q} |T^{k,2}(f)(x)|^s dx\right)^{1/s}\left(\int_{2Q} |b(x)-b_Q|^{sr/(s-r)} dx\right)^{(s-r)/sr} \\
\leq & C||b||_{BMO}\left(\frac{1}{|Q|}\int_{2Q} |T^{k,2}(f)(x)|^s dx\right)^{1/s} \\
\leq & C||b||_{BMO}M_s(T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

thus

$$\begin{aligned}
I_5 & \leq \sum_{k=1}^{l}\frac{1}{|Q|}\int_Q ||F_t^{k,1}M_{(b-b_Q)\chi_{2Q}}F_t^{k,2}(f)(x)||dx \\
& \leq C||b||_{BMO}\sum_{k=1}^{m} M_s(T^{k,2}(f))(\tilde{x}).
\end{aligned}$$

For I_6, recalling that $s > q'$, taking $1 < p < \infty, 1 < r < s$ with $1/p + 1/q + 1/r = 1$, by the boundedness of T, we get, for $x \in Q$,

$$\begin{aligned}
& ||F_t^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}F_t^{k,2}(f)(x) - F_t^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}F_t^{k,2}(f)(x_0)|| \\
\leq & \int_{(2Q)^c} |b(y)-b_{2Q}|||F_t(x,y)-F_t(x_0,y)|||T^{k,2}(f)(y)|dy \\
\leq & \sum_{j=1}^{\infty}\int_{2^jd\leq|y-x_0|<2^{j+1}d} ||F_t(x,y)-F_t(x_0,y)|||b(y)-b_{2Q}||T^{k,2}(f)(y)|dy \\
\leq & \sum_{j=1}^{\infty}\left(\int_{2^jd\leq|y-x_0|<2^{j+1}d} ||F_t(x,y)-F_t(x_0,y)||^q dy\right)^{1/q} \\
& \times\left(\int_{2^{j+1}Q} |b(y)-b_Q|^p dy\right)^{1/p}\left(\int_{2^{j+1}Q} |T^{k,2}(f)(y)|^r dy\right)^{1/r} \\
\leq & C||b||_{BMO}\sum_{j=1}^{\infty} C_j(2^jd)^{-n/q'}j(2^jd)^{n/p}(2^jd)^{n/s}\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q} |T^{k,2}(f)(y)|^s dy\right)^{1/s} \\
\leq & C||b||_{BMO}M_s(T^{k,2}(f))(\tilde{x})\sum_{j=1}^{\infty} jC_j \\
\leq & C||b||_{BMO}M_s(T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

thus

$$\begin{aligned} I_6 &\le \frac{1}{|Q|}\int_Q \sum_{k=1}^{l} ||F_t^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} F_t^{k,2}(f)(x) - F_t^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} F_t^{k,2}(f)(x_0)||dx \\ &\le C||b||_{BMO}\sum_{k=1}^{l} M_s(T^{k,2}(f))(\tilde{x}). \end{aligned}$$

This completes the proof of Theorem 2.3. □

Proof. Of Theorem 2.4. Choose $q' < s < p$ in Theorem 2.1, we have, by Lemma 3.1, 3.3 and 3.4,

$$\begin{aligned} &||T^b(f)||_{L^r} \le \|M(T^b(f))\|_{L^r} \le C\|M^{\#}(T^b(f))\|_{L^r} \\ \le\ & C||b||_{Lip_\beta}\sum_{k=1}^{m}\|M_{\beta,s}(T^{k,2}(f))\|_{L^r} \le C||b||_{Lip_\beta}\sum_{k=1}^{m}\|T^{k,2}(f)\|_{L^p} \\ \le\ & C||b||_{Lip_\beta}\|f\|_{L^p}. \end{aligned}$$

This completes the proof. □

Proof. Of Theorem 2.5. Choose $q' < s < p$ in Theorem 2.1, we have, by Lemma 3.5 and 3.6,

$$\begin{aligned} &||T^b(f)||_{L^{r,\varphi}} \le \|M(T^b(f))\|_{L^{r,\varphi}} \le C\|M^{\#}(T^b(f))\|_{L^{r,\varphi}} \\ \le\ & C||b||_{Lip_\beta}\sum_{k=1}^{m}\|M_{\beta,s}(T^{k,2}(f))\|_{L^{r,\varphi}} \le C||b||_{Lip_\beta}\sum_{k=1}^{m}\|T^{k,2}(f)\|_{L^{p,\varphi}} \\ \le\ & C||b||_{Lip_\beta}\|f\|_{L^{p,\varphi}}. \end{aligned}$$

This completes the proof. □

Proof. Of Theorem 2.6. Choose $q' < s < p$ in Theorem 2.2, we have, by Lemma 3.1, 3.2 and 3.3,

$$\begin{aligned} &||T^b(f)||_{\dot{F}_p^{\beta,\infty}} \le C\left|\left|\sup_{Q\ni\cdot}\frac{1}{|Q|^{1+\beta/n}}\int_Q\left|T^b(f)(x)-C_0\right|dx\right|\right|_{L^p} \\ \le\ & C||b||_{Lip_\beta}\sum_{k=1}^{m}\|M_s(T^{k,2}(f))\|_{L^p} \le C||b||_{Lip_\beta}\sum_{k=1}^{m}\|T^{k,2}(f)\|_{L^p} \\ \le\ & C||b||_{Lip_\beta}||f||_{L^p}. \end{aligned}$$

This completes the proof. □

Proof. Of Theorem 2.7. Choose $q' < s < p$ in Theorem 2.3, we have, by Lemma 3.1, 3.3 and 3.4,

$$\begin{aligned} &||T^b(f)||_{L^p} \le \|M(T^b(f))\|_{L^p} \le C\|M^{\#}(T^b(f))\|_{L^p} \\ \le\ & C||b||_{BMO}\sum_{k=1}^{m}\|M_s(T^{k,2}(f))\|_{L^p} \le C||b||_{BMO}\sum_{k=1}^{m}\|T^{k,2}(f)\|_{L^p} \\ \le\ & C||b||_{BMO}\|f\|_{L^p}. \end{aligned}$$

This completes the proof. □

Proof. Of Theorem 2.8. Choose $q' < s < p$ in Theorem 2.3, we have, by Lemma 3.5 and 3.6,

$$\begin{aligned} & ||T^b(f)||_{L^{p,\varphi}} \le \|M(T^b(f))\|_{L^{p,\varphi}} \le C\|M^{\#}(T^b(f))\|_{L^{p,\varphi}} \\ \le & \quad C||b||_{BMO}\sum_{k=1}^m \|M_s(T^{k,2}(f))\|_{L^{p,\varphi}} \le C||b||_{BMO}\sum_{k=1}^m \|T^{k,2}(f)\|_{L^{p,\varphi}} \\ \le & \quad C||b||_{BMO}\|f\|_{L^{p,\varphi}}. \end{aligned}$$

This completes the proof. □

4. Applications

In this section we shall apply the Theorems 2.1-2.8 of the paper to some particular operators such as the Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

Application 4.1. Littlewood-Paley operator.

Fixed $\varepsilon > 0$. Let ψ be a fixed function which satisfies:

(1) $\int_{R^n} \psi(x)dx = 0$,

(2) $|\psi(x)| \le C(1+|x|)^{-(n+1)}$,

(3) $|\psi(x+y)-\psi(x)| \le C|y|^{\varepsilon}(1+|x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$;

Let $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$ and $F_t(f)(x) = \int_{R^n} f(y)\psi_t(x-y)dy$. The Littlewood-Paley operator is defined (see [23])

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2\frac{dt}{t}\right)^{1/2}.$$

Set H be the space

$$H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t\right)^{1/2} < \infty\right\}.$$

Let b be a locally integrable function on R^n. The Toeplitz type operator related to the Littlewood-Paley operator is defined by

$$g^b_\psi(f)(x) = \left(\int_0^\infty |F^b_t(f)(x)|^2\frac{dt}{t}\right)^{1/2},$$

where

$$F^b_t = \sum_{k=1}^m F^{k,1}_t M_b F^{k,2}_t,$$

$F^{k,1}_t$ are F_t or $\pm I$(the identity operator), $T^{k,2} = ||F^{k,2}_t||$ are the bounded linear operators on $L^p(R^n)$ for $1 < p < \infty$ and $k = 1, ..., m$, $M_b(f) = bf$. Then, for each fixed $x \in R^n$, $F^b_t(f)(x)$ may be viewed as the mapping from $[0, +\infty)$ to H, and it is clear that

$$g^b_\psi(f)(x) = ||F^b_t(f)(x)||, \quad g_\psi(f)(x) = ||F_t(f)(x)||.$$

It is easily to see that g^b_ψ satisfies the conditions of Theorems 2.1-2.8 (see [12-14]), thus Theorems 2.1-2.8 hold for g^b_ψ.

Application 4.2. Marcinkiewicz operator.

Fixed $0 < \gamma \le 1$. Let Ω be homogeneous of degree zero on R^n with $\int_{S^{n-1}} \Omega(x')d\sigma(x')$ $= 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$. Set $F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy$. The

Marcinkiewicz operator is defined by (see [24])

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2}.$$

Set H be the space

$$H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t^3\right)^{1/2} < \infty\right\}.$$

Let b be a locally integrable function on R^n. The Toeplitz type operator related to the Marcinkiewicz operator is defined by

$$\mu_\Omega^b(f)(x) = \left(\int_0^\infty |F_t^b(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_t^b = \sum_{k=1}^m F_t^{k,1} M_b F_t^{k,2},$$

$F_t^{k,1}$ are F_t or $\pm I$(the identity operator), $T^{k,2} = ||F_t^{k,2}||$ are the bounded linear operators on $L^p(R^n)$ for $1 < p < \infty$ and $k = 1, ..., m$, $M_b(f) = bf$. Then, it is clear that

$$\mu_\Omega^b(f)(x) = ||F_t^b(f)(x)||, \quad \mu_\Omega(f)(x) = ||F_t(f)(x)||.$$

It is easily to see that μ_Ω^b satisfies the conditions of Theorems 2.1-2.8 (see [12-14][24]), thus Theorems 2.1-2.8 hold for μ_Ω^b.

Application 4.3. Bochner-Riesz operator.

Let $\delta > (n-1)/2$, $F_t^\delta(\hat{f})(\xi) = (1 - t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_t^\delta(z) = t^{-n} B^\delta(z/t)$ for $t > 0$. The maximal Bochner-Riesz operator is defined by (see [15])

$$B_{\delta,*}(f)(x) = \sup_{t>0} |F_t^\delta(f)(x)|.$$

Set H be the space $H = \{h : ||h|| = \sup_{t>0} |h(t)| < \infty\}$. Let b be a locally integrable function on R^n. The Toeplitz type operator related to the maximal Bochner-Riesz operator is defined by

$$B_{\delta,*}^b(f)(x) = \sup_{t>0} |B_{\delta,t}^b(f)(x)|,$$

where

$$B_{\delta,t}^b = \sum_{k=1}^m F_t^{k,1} M_b F_t^{k,2},$$

$F_t^{k,1}$ are F_t or $\pm I$(the identity operator), $T^{k,2} = ||F_t^{k,2}||$ are the bounded linear operators on $L^p(R^n)$ for $1 < p < \infty$ and $k = 1, ..., m$, $M_b(f) = bf$. Then

$$B_{\delta,*}^b(f)(x) = ||B_{\delta,t}^b(f)(x)||, \quad B_*^\delta(f)(x) = ||B_t^\delta(f)(x)||.$$

It is easily to see that $B_{\delta,*}^b$ satisfies the conditions of Theorems 2.1-2.8 (see [12][13]), thus Theorems 2.1-2.8 hold for $B_{\delta,*}^b$.

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